Nonequilibrium Phase Transitions Induced by Multiplicative Noise
— Creation of an Ordered Phase by Noise! —

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Noise is usually a source of disorder. However, the same noise can generate an ordered state in nonlinear nonequilibrium systems. We demonstrates the existence of such surprising phenomena, noise-induced nonequilibrium phase transitions, using a simple mathematical model and computer simulations. The critical properties of these phase transitions appeared to be very similar to those of equilibrium phase transtions and compatible with those of the dynamical Landau-Ginzburg model.

Noise is usually thought of as a phenomenon which perturbs the observation and creates disorder. This idea is based mainly on our day to day experience. Noise indeed disturbs equilibrium systems and drives them into disordered states. The effect of noise can, however, be quite different in nonlinear nonequilibrium systems. Several situations have been documented in the literature, in which the noise actually participates in the creation of ordered states or is responsible for surprising new phenomena through its interaction with the nonlinearities of the system [1–3].

Recently, a novel and quite spectacular phenomenon was discovered in a specific model of a spatially distributed system with multiplicative noise, white in space and time [4]. It was found that the noise generates an ordered symmetry-breaking state through a genuine second-order phase transition whereas no such transition is observed in the absence of noise. In this report, we briefly review our recent computer simulations of such noise-induced nonequilibrium phase transitions.

Before going to a spatially extended system, let us look at a simple zero-dimension model given by the following stochastic differential equation:

$$\dot{x} = f(x) + g(x)\xi \tag{1}$$

where x is a stochastic variable and ξ stands for Gaussian white noise. A nonlinear function $f(x)$ defines a deterministic time evolution and the other nonlinear function $g(x)$ introduces a coupling with the noise. Using an appropriate choice of $f(x)$, the solution reaches to a steady state in which x fluctuates around a mean value $\bar{x}$. We are interested in how $\bar{x}$ and its fluctuation Δx depend on a control parameter σ, which is in our case the strength of noise defined by

$$\langle \xi(t)\xi(t') \rangle = \sigma^2 \delta(t-t'). \tag{2}$$

Eq. (1) is interpreted according to the Stratonovitch interpretation. [5] Hence, the probability density $P(x,t)$ for the variable $x(t)$ obeys the following Fokker-Planck equation [6]:

$$\partial_t P(x,t) = -\partial_x[f(x)P(x,t)] + \frac{\sigma^2}{2}\partial_x \left\{g(x)\partial_x[g(x)P(x,t)]\right\} \tag{3}$$

This problem can be solved analytically and $\bar{x}$ obeys the following equation :

$$f(\bar{x}) - \frac{\sigma^2}{2} g(\bar{x})g'(\bar{x}) = 0. \tag{4}$$

where $g'(x)$ stands for the derivative of $g(x)$ with respect to x. The steady states in the absence of noise are determined by $f(\bar{x}) = 0$ which is different from Eq. (4). Therefore, the steady state solutions of Eq. (1) could be quite different from the dterministic systems. These changes in asymptotic behavior of the system have been generally named noise-induced transitions.

To illustrate this phenomenon, consider the case of a deterministically stable steady state at $x = 0$, e.g. $f(x) = -x(1+x^2)^2$ perturbed by a multiplicative noise $g(x) = 1-x^2$. Then, Eq. (4) can have multiple solutions. Using a linear stability analysis, we found that $\bar{x}$ bifurcates at a critical strength of noise $\sigma_c = 2$ in this example case. In another word, there is only one solution $\bar{x} = 0$ for $\sigma^2 < 2$ but there are two solutions for $\sigma^2 > 2$; one is positive and the other negetive. Choice of either sign is equally probable but the system must choose one solution. Therefore, the symmetry is broken. Note that if $g(x) = 1+x^2$ is used, there is only one solution regardless the strength of noise and thus no noise-induced transition occurs.

Next, we consider a d-dimension lattice with a scalar stochastic variable x_i at the ith site. The time evolution of x_i is described by a set of coupled stochastic equations

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i + \frac{D}{2d}\sum_{j\in N(i)}(x_i - x_j) \qquad (5)$$

where D denotes a strength of spatial coupling and the summation is taken over the nearest neighbors. We focus on an order parameter $\bar{x} = \sum_i^N x_i/N$ which represents the spatial order of the system.

Unfortunately, an analytic solution to this problem is not known. So, let us try to guess a solution. When the spatial coupling is weak, the each site is independent and evolves in the same way as the zero-dimensional model. Therefore, all x_i goes to a steady state $\bar{x} = 0$ below the critical point. Above the critical point x_i takes one of the bifurcated states. Since the choice of the branches is at random, $\bar{x} = 0$ regardless the strength of noise. On the other hand, in the strong coupling limit, the coupling forces x_i to take a similar value to those of the nearest neighbors. Then, above the critical noise all sites are expected to choose the same branch of bifurcated solutions. If it happens, a spatially ordered state ($\bar{x} \neq 0$) is formed.

Surprisingly, such an intuitive argument appeared to be wrong. Using the same $f(x)$ and $g(x)$ as in the zero dimension model, no ordered phase was found. We found that in the strong coupling limit the mean value $\bar{x}$ must obey a similar condition to the zero dimension model:

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma^2}{2}g(\bar{x})g'(\bar{x}) \qquad (6)$$

However, the sign of the multiplicative noise term is **opposite** to that of Eq. (4)! The condition of bifurcation in spatially extended systems is precisely opposite to that in zero dimension model. From Eq. (6), we expect that a pure noise-induced nonequilibrium phase transition occurs if $g(x) = 1 + x^2$ is used for which no transition is found in the zero dimension model.

In order to demonstrate the existence of such noise-induced nonequilibrium phase transitions and to determine the critical properties of such transitions, we performed extensive computer simulations using 2D square lattices. The coupling constant $D = 20$ is sufficiently large to cause the transitions. Unlike equilibrium systems, nonequilibrium systems are time-dependent. We must integrate a large set of coupled stochastic differential equations for a sufficiently long period to obtain accurate statistics, which demands very fast computers. Cray C90 at the Alabama Supercomputer Center was used for most simulations but a massively parallel computer Connection Machine model 5 was needed for the large lattices. The detailed numerical algorithms we employed are explained in Appendix.

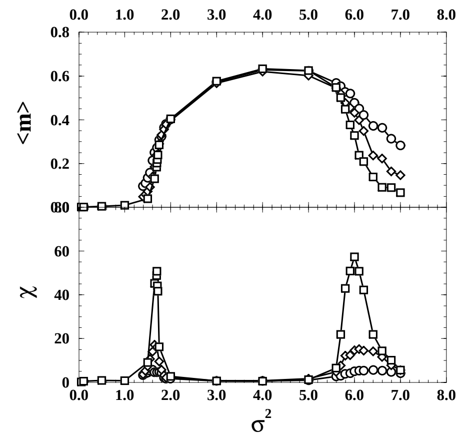

FIG. 1. *Upper curve:* Order parameter $< x >$ versus intensity of the noise for system size 32×32 (circles), 64×64 (diamonds) and 128×128 (squares). Notice that although the general features of mean field approximations agree with the simulation result, they tend to overestimate the ordered region. *Lower curve:* Susceptibility, $\chi = \frac{L^2}{\sigma^2}\left[\langle x^2\rangle - \langle x\rangle^2\right]$, as a function of σ^2 The peaks cleary show the enhancement of fluctuations around the two critical points.

In figure 1, the order parameter and susceptibility as functions of noise intensity are plotted for three difference lattice sizes. An ordered phase first appears as the noise intensity increases to $\sigma^2 = 1.51$ and then disappears again at $\sigma^2 = 5.8$ (reentrant transition). The finite size scaling method is used to find the precise critical points. Susceptibility has clear peaks at the critical points. Both spatial and temporal correlation lengths diverge also at the critical points. All these critical properties are very similar to those of equilibrium phase transitions. Furthermore, the finite size scaling suggests that the critical properties of both first and reentrant transitions are compatible with those of the 2D Ising universality class.

Snapshots of the field values at several different noise intensities are shown in figure 2. A weak random pattern ($\sigma = 0.50$) is observed below the first critical point. It is clear that the field simply fluctuates around $\bar{x} = 0$. At $\sigma = 1.72$, many islands with different sizes (fractal structure) appear, indicating that the system is in the critical region. Betnween two transition points, the whole area is covered by positive values except for several small spots($\sigma = 4.00$). This is an ordered phase. At $\sigma = 5.80$, islands with diferent scales again appear and the ordered phase begins to be destroyed. Finally, at $\sigma = 8.00$, the field values change from large negative values to positive values and they are almost randomly distributed. Although shot-range ordered is still present, the long-range spatial order is completely destroyed.

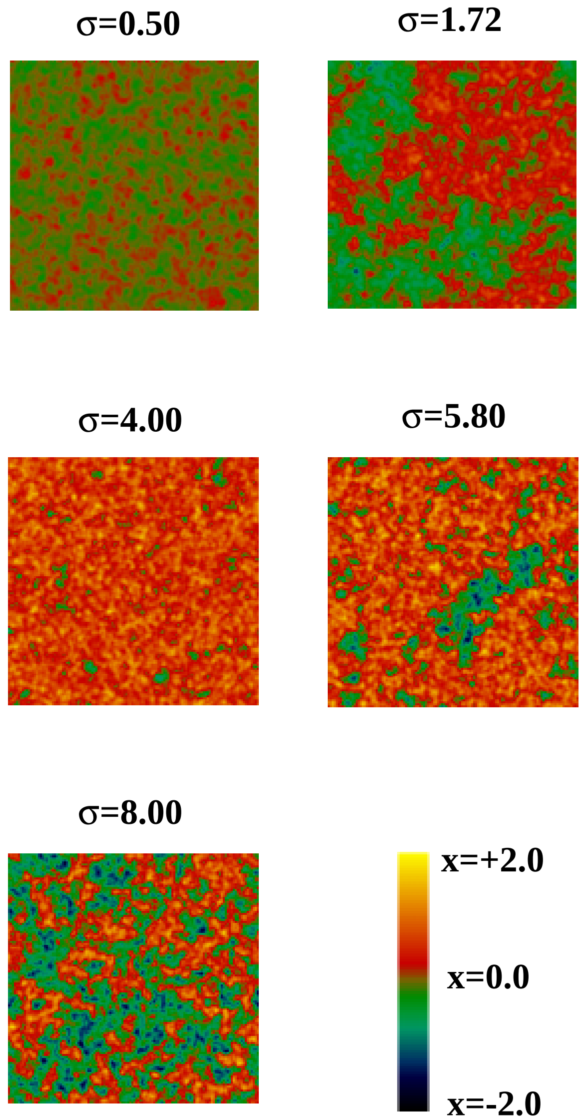

FIG. 2. Snapshots of the field values on a 128×128 lattice. Green to blue color corresonds to negative values and red to yellow positive values. Field values are almost random below the first critical point and above the second critical point ($\sigma = 0.50$ and $\sigma = 8.00$. Many islands with different sizes are observed at the critical points ($\sigma = 1.72$ and $\sigma = 5.80$). Between the critical points, an ordered phase is obtained ($\sigma = 4.00$)

In figure 3, we illustrate the growth of ordered phase. Dark areas correspond to negative values of the field x_i and light areas to positive values. Initially, the system was in a random configuration. Many small ordered regions are quickly developed and as time evolve the ordered areas grow. After a while, two large ordered domains; one with positive values and the other with the opposite sign, are formed. Eventually, one of the domains will grow and the other will disappear. When the syswtem reaches a steady state, the whole area is covered by a single domain.

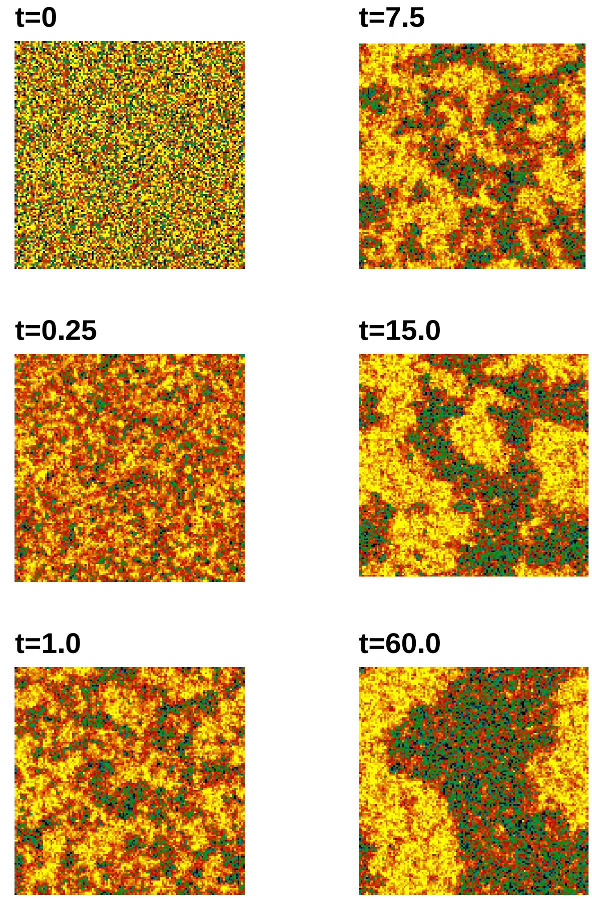

FIG. 3. Time evolution of domains starting in a completely random initial configuration toward an ordered phase for the spatially extended model on a square lattice, ($L = 128$, $\sigma^2 = 4.00$ and $D = 20$). Dark areas correspond to negative values of the field x_i and light areas to positive values. Notice the initial development of small ordered regions which subsequently grow.

In conclusion, we have confirmed the existence of a pure noise-induced nonequilibrium phase transition using a simple model. We have obtained evidence that the critical properties of the transition are compatible with those of Ising universality class. It is clear that phase transition of other kinds, e.g., first order transition and transitions that break both temporal and spatial symmetries simultaneously are possible through similar multiplicative noise.

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APPENDIX: COMPUTER SIMULATIONS

A Monte Carlo simulation of the stochastic process (5) was performed for 2-dimensional square lattices of various sizes up to $L = 128$ with periodic boundary conditions. The stochastic differential equation for the variable at the i-th site, x_i, is given by

$$\frac{dx_i}{dt} = F_i(\mathbf{x}) + G_i(\mathbf{x})\xi_i(t), \qquad i = 1, \ldots, N = L^2 \quad \text{(A1)}$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ and

$$F_i(\mathbf{x}) = f(x_i) - \frac{D}{4} \sum_{j \in n(i)} (x_i - x_j) \tag{A2}$$

$$G_i(\mathbf{x}) = g(x_i) \tag{A3}$$

This equation was integrated using two different algorithms, the Milshtein and the Heun methods [7,8].

The Milshtein method allows to advance forward in time by means of the recursion relations:

$$x_i(t+\delta t) = \left[F_i(\mathbf{x}(t)) + \frac{\sigma^2}{2} G_i(\mathbf{x}(t)) \frac{dG_i(\mathbf{x}(t))}{dx_i} \right] \delta t \tag{A4}$$

$$+ G_i(\mathbf{x}(t)) \sqrt{\sigma^2 \delta t} \eta_i(t) \tag{A5}$$

where $\eta_i(t)$ are independent Gaussian random variables of zero mean and variance equal to 1 and the second term is included because (A1) is interpreted in the Stratonovich sense. The order of numerical error in the Milshtein method is δt. Therefore, a small δt (e.g., $\delta t = 1 \times 10^{-4}$ for $\sigma^2 = 1$) must be used, while its computational effort per time step is relatively small. For large σ, where fluctuations are rapid and large, a longer integration period and a smaller δt is necessary. The Milshtein method quickly becomes impractical.

The Heun method is based on the 2nd-order Runge-Kutta method and integrates the stochastic equation by a recursive equation

$$x_i(t+\delta t) = x_i(t) + \frac{\delta t}{2} \left[F_i(\mathbf{x}(t)) + F_i(\mathbf{y}(t)) \right] \tag{A6}$$

$$+ \frac{\sqrt{\sigma^2 \delta t}}{2} \eta_i(t) \left[G_i(\mathbf{x}(t)) + G_i(\mathbf{y}(t)) \right] \tag{A7}$$

where

$$y_i(t) = x_i(t) + f(x_i(t))\delta t + g(x_i(t))\eta_i(t)\sqrt{\sigma^2 \delta t} \tag{A8}$$

This method allows larger δt than the Milshtein method without significant increase in computational effort per step. We used this method for $\sigma^2 > 2$.

The time step δt has been chosen by a stability condition and also such that averaged magnitudes do not depend on δt within statistical errors. For $D = 20$, for example, the necessary values for δt vary between $\delta t = 5 \times 10^{-4}$ for $\sigma^2 = 1$ and $\delta t = 1 \times 10^{-5}$ for $\sigma^2 = 15$. The Gaussian random numbers necessary for the simulations were generated either by using the Box-Muller-Wiener algorithm or a very fast numerical inversion method [9]. The time evolution of the average value is carefully monitored until the stationary state is reached.

The order parameter is computed by

$$\langle m \rangle = \left\langle \left\langle \mid \frac{1}{L^2} \sum_{i=1}^{N} x_i \mid \right\rangle_{\rm time} \right\rangle_{\rm ensemble} \tag{A9}$$

where $\langle \ldots \rangle_{\rm time}$ and $\langle \ldots \rangle_{\rm ensemble}$ indicate time average and ensemble average, respectively. The averaging time T was chosen to be sufficiently longer than the correlation time, for example, $T \approx 2 \times 10^4$ (10^8 steps) near the critical points. The ensemble average was taken over at least 10 independent systems. Similarly, the susceptibility is evaluated as

$$\chi = \frac{L^2}{\sigma^2} \left\langle \left\langle \left(\frac{1}{L^2} \sum_{i=1}^{N} x_i \right)^2 - \langle m \rangle^2 \right\rangle_{\rm time} \right\rangle_{\rm ensemble} . \tag{A10}$$

Simulation of large systems (128×128) was too long for Cray C90 despite the code is mostly vectorized. Therefore, we used a massively parallel computer, the connection machine model 5E with 256 processors which appeared to be about ten times faster than Cray C90 for this particular application with the same programs.

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[5] There exist an infinity of interpretations of stochastic differential equations with multiplicative noise, with, on one extreme, the mathematically minded Itô rule (corresponding to a diffusion term of the form $\partial_x^2[g^2(x)P(x,t)]$), and on the other extreme the kinetic definition (corresponding to a diffusion term of the form $\partial_x[g^2(x)\partial_x P(x,t)]$). Which interpretation is valid depends on the origin of the white noise and its interaction with eventual other fast degrees of freedom of the system.

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