Macroscopic limit cycle via a noise-induced phase transition

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ABSTRACT

Bistability generated via a noise-induced phase transition is reexamined from the view of macroscopic dynamical systems, which clarifies the role of fluctuation better than the conventional Fokker-Plank or Langevin equation approach. Using this approach, we investigated the spatially-extended systems with two degrees of freedom per site. The model systems undergo a noise-induced phase transition through a Hopf bifurcation, leading to a macroscopic limit cycle motion similar to the deterministic relaxation oscillation.

Keywords: nonequilibrium phase transition, noise-induced phase transition, nonlinear dynamical systems, stochastic processes, Hopf bifurcation, pitchfork bifurcation

1. INTRODUCTION

Interplay between nonlinear dynamics and noise often generates interesting phenomena such as stochastic resonance[1] in which noise plays a constructive role. Another example is a pure noise-induced phase transition[2–4] in spatially extended systems where the ordered phase is absent without noise. Unlike noise-induced transition in the systems of small degrees of freedom,[5] the noise-induced *phase* transition breaks ergodicity and behaves like equilibrium phase transitions. The pioneering work by Van den Broeck *et al.*[2,3] discussed a pitchfork bifurcation leading to a macroscopically bistable state. In addition to the second order transition to an ordered phase, they found a reentrant transition at a higher noise intensity. Since then, many variations of pure noise-induced phase transitions were introduced.[4] Spatial patterns can be induced via the noise-induced phase transition.[6,7] The noise-induced first order phase transition was also shown to be possible.[8–10] The systems with colored noise were investigated by various groups.[10–12] Furthermore, the bistability created by the noise-induced phase transition exhibits stochastic resonance when a time-periodic perturbation is added.[13] This result suggests that the bistability is stable against external perturbation despite that it is purely generated by noise. Similarly, it should be possible to rock the noise-induced "bistable potential" by adding another degree of freedom, reminiscent to the relaxation oscillation in deterministic bistable systems. The main purpose of the present paper is to investigate such a noise induced macroscopic oscillation.

All previous works begin with a nonlinear mesoscopic equation, either Langevin or Fokker-Plank equation. Then, a macroscopic order parameter is introduced as a mean value of microscopic dynamical variables. As van Kampen pointed out,[14] the dynamics of such a macroscopic variable cannot be determined without knowing the dynamics of internal fluctuation owing to the nonlinearity of the systems, which makes it difficult to interpret the dynamics of macroscopic quantities. In this paper, we attempt to describe the instability of macroscopic quantities using macroscopic equations of motion. Although such approach is mathematically less tractable than standard methods, it provides the relation between macroscopic dynamics and its internal fluctuation, which plays a key role in noise-induced phase transition. As examples, we investigate two systems: one with a single macroscopic variable that undergoes a pitchfork bifurcation equivalent to the previously investigated pure noise-induced phase transition (Section 2) and the other with two macroscopic variables that undergoes a Hopf bifurcation leading to a macroscopic limit cycle motion (Section 3).

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2. NOISE-INDUCED MACROSCOPIC PITCHFORK BIFURCATION

2.1. The Model

Consider the following globally-coupled model

$$\dot{x_i} = f(x_i) - \frac{D}{N}\sum_{j=1}^{N}(x_i - x_j) + g(x_i)\xi_i(t) \tag{1}$$

which was originally investigated by Van den Broeck *et al.*.[2,3] In Eq. (1) N and D are the number of sites and coupling strength, respectively. A Gaussian white noise, $\xi_i(t)$ is defined by

$$\langle \xi_i(t) \rangle = 0, \qquad \langle \xi_i(t)\xi_j(t') \rangle = \sigma^2 \delta_{ij}\delta(t-t') \tag{2}$$

Equation (1) is interpreted in the Stratonovich sense. We are interested in the dynamics of a macroscopic quantity, $\langle x \rangle = \frac{1}{N}\sum_i x_i$, and the mean value at a stationary state, $\langle x \rangle^*$ which is used as an order parameter.

For a noiseless or weak noise case, $\langle x \rangle^*$=0 is a stable fixed point. It has been shown that when the noise intensity is increased, the fixed point becomes unstable and increasing the noise intensity further, the fixed point becomes stable again (reentrant transition). Nonlinear functions,

$$f(x) = -x(1+x^2)^2, \qquad g(x) = 1 + x^2 \tag{3}$$

are known to produce such a pure noise-induced phase transition and reentrant transition.[2–4]

In previous works, this system is investigated by a standard mean-field procedure replacing Eq. (1) with

$$\dot{x} = f(x) - D(x - \langle x \rangle) + g(x)\xi(t) \tag{4}$$

where $\langle x \rangle$ is also replaced with a mean field defined by a self-consistent equation

$$\langle x \rangle = \int_{-\infty}^{+\infty} xP(x, \langle x \rangle)dx \tag{5}$$

using a stationary state probability distribution $P(x, \langle x \rangle)$. We assume that the stationary distribution is uniquely determined for a given vale of $\langle x \rangle$ from the corresponding Fokker-Plank equation. The phase boundary in the σ^2-D parameter space can be implicitly obtained by taking derivative of Eq. (5) with respect to $\langle x \rangle$. While this approach provides an exact phase boundary, the origin of the phase transition, in particular the cause of the reentrant transition is not clear. Furthermore, a standard bifurcation theory for noisy dynamical systems is desired for the comparison with similar deterministic dynamical systems. The following approximate approach provides a useful physical insight for the noisy dynamical systems.

2.2. Macroscopic Dynamics and Moment expansion

Instead of resorting to a Fokker-Plank equation, we try to express the dynamics of macroscopic quantities as a set of ordinary differential equations. From Eq. (4), the dynamics of the mean $\langle x \rangle$ follows

$$\langle \dot{x} \rangle = \langle f(x) \rangle + \frac{\sigma^2}{2}\langle g'(x)g(x) \rangle . \tag{6}$$

While this expression is exact, it is not useful since $\langle f(x) \rangle$ and $\langle g'(x)g(x) \rangle$ are not known. In the previous papers, the origin of instability was investigated by an approximate dynamics

$$\begin{aligned} \langle \dot{x} \rangle &\approx f(\langle x \rangle) + \frac{\sigma^2}{2} g'(\langle x \rangle)g(\langle x \rangle) \\ &= (\sigma^2 - 1)\langle x \rangle + (\sigma^2 - 2)\langle x \rangle^3 - \langle x \rangle^5 \end{aligned} \tag{7}$$

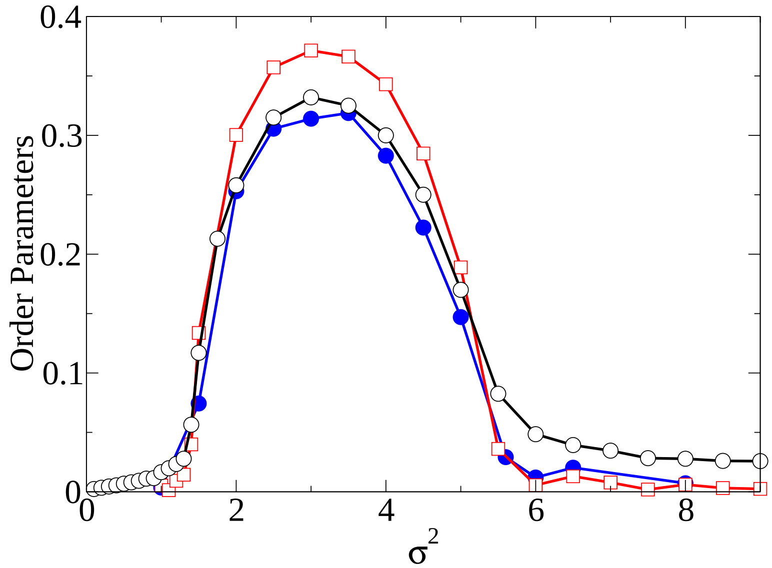

Figure 1. Order parameters $\langle x \rangle$ for pitchfork bifurcation (square) and r for Hopf bifurcation (circle). Coupling constants $D = 10$ and $k = 0.1$ are used. For the Hopf bifurcation two different system sizes, N=10000 and N=200000, to check finite size effect. The smaller system shows a long tail above $\sigma^2 = 6$ due to the finite size effect. All three lines show the same critical values, $\sigma^2_{c,1} \approx 1.3$ and $\sigma^2_{c,2} \approx 5.5$.

assuming the fluctuation $\Delta \equiv x - \langle x \rangle$ is negligible. This approximation is valid for strong coupling (exact for $K \to +\infty$). A linear stability analysis immediately tells that the fixed point $\langle x \rangle^* = 0$ becomes unstable for $\sigma^2 > 1$. However, Eq. (7) fails to predict the existence of reentrant transition.

In order to go beyond the zeroth order approximation, we expand Eq. (6) in terms of moments $\mu_n = \langle \Delta^n \rangle$ as follows:

$$\begin{aligned}
\langle \dot{x} \rangle &= \sum_{n=0} \frac{\mu_n}{n!} \left\{ f^{(n)}(\langle x \rangle) + \frac{\sigma^2}{2} [g'(\langle x \rangle) g(\langle x \rangle)]^{(n)} \right\} \\
&= (\sigma^2 - 2)\mu_3 - \mu_5 + \left[\sigma^2 - 1 + 3(\sigma^2 - 2)\mu_2 - 5\mu_4\right] \langle x \rangle - 10\mu_3 \langle x \rangle^2 + (\sigma^2 - 2 - 10\mu_2)\langle x \rangle^3 - \langle x \rangle^5 \qquad (8)
\end{aligned}$$

where $f^{(n)}$ indicates n-th order derivative.[14] This equation does not show an explicit dependency on the coupling strength D. The effect of the coupling enters via the equations of motion for the moments:

$$\begin{aligned}
\dot{\mu}_n &= -Dn\mu_n + \sum_{m=0} \frac{n\mu_{n+m-1}}{m!} \left\{ f^{(m)}(\langle x \rangle) + \frac{\sigma^2}{2} [g'(\langle x \rangle) g(\langle x \rangle)]^{(m)} \right\} \\
&+ \sum_{m=0} \frac{n(n-1)\mu_{n+m-2}}{m!} \frac{\sigma^2}{2} [g^2(\langle x \rangle)]^{(m)} \qquad (9)
\end{aligned}$$

where $\mu_0 = 1$ and $\mu_1 = 0$. For the lowest order moment Eq. (9) becomes

$$\begin{aligned}
\dot{\mu_2} &= \sigma^2 + 2[2\sigma^2 - (1 + D)]\mu_2 + \left(3\sigma^2 - 4\right)\mu_4 - 2\mu_6 + \left[2(5\sigma^2 - 6)\mu_3 - 10\mu_5\right] \langle x \rangle \\
&+ \left[2\sigma^2 + 12(\sigma^2 - 1)\mu_2 - 20\mu_4\right] \langle x \rangle^2 - 20\mu_3 \langle x \rangle^3 + (\sigma^2 - 10\mu_2)\langle x \rangle^4 \qquad (10)
\end{aligned}$$

Since the nonlinear Langevin equation Eq. (1) is not a Gaussian process, all moments must be taken into account.[14] Therefore, one has to solve infinite simultaneous equations to obtain exact solutions. However, certain qualitative features of the solutions can be obtained without their exact expressions. Furthermore, one can find approximate solutions that qualitatively capture the properties of noise induced transition.

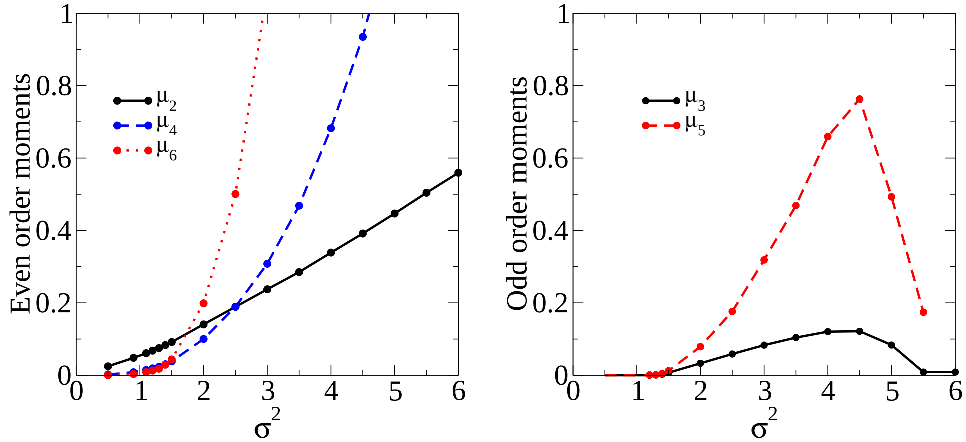

Figure 2. Stationary moments μ_n^* obtained by numerical simulation for D=10 and $N = 10000$. Even order moments (left panel) increases monotonically and no singular behavior can be seen at critical points. Higher order moments grow faster than the 2nd order moment which causes the reentrant transition. On the other hand, odd order moments (right panel) show the same type of bifurcation as $\langle x \rangle$.

Let $\langle x \rangle^*$ and μ_n^* be a stationary solution to Eqs. (8) and (9). Since the system has a reflection symmetry with respect to $x = 0$, there is a fixed point with $\langle x \rangle^* = \mu_{2n+1}^* = 0$. Since μ_{2n}^* are all positive, there is always a symmetric fluctuation under the presence of noise. We are interested in the stability of this fixed point. $\langle x \rangle^*$ or μ_{2n+1}^* servers as an order parameter and σ^2 and D are the control variables.

Since the system has infinite degrees of freedom, there could be many directions in which the solution is unstable. First, we focus on the stability along $\langle x \rangle$ by fixing all moments at their fixed points. Evolution of a small displacement $\epsilon = \langle x \rangle - \langle x \rangle^*$ follows

$$\dot{\epsilon} = \left[\sigma^2 - 1 + 3(\sigma^2 - 2)\mu_2^* - 5\mu_4^*\right]\epsilon + (\sigma^2 - 2 - 10\mu_2^*)\epsilon^3 - \epsilon^5 \tag{11}$$

Linear stability analysis finds that the fixed point at $\langle x \rangle^*$=0 becomes unstable via a pitchfork bifurcation. The critical point is determined by

$$(\sigma_c^2 - 1) + 3(\sigma_c^2 - 2)\mu_2^* - 5\mu_4^* = 0 \quad . \tag{12}$$

Since the moments depend on σ^2, we cannot find the critical noise intensity from this relation alone. However, usefull information can be deducted from it without detailed knowledge of the moments. Noting that both μ_2^* and μ_4^* are non-negative, the L. H. S. of Eq. (12) is negative for $\sigma^2 < 1$ regardless of the magnitude of μ_2^* and μ_4^*. Therefore, $\langle x \rangle^* = 0$ is always a stable fixed point when $\sigma^2 < 1$. For $1 < \sigma^2 < 2$, only the 0th moment term is positive. The higher moment terms are both negative and against instability. Therefore, $\sigma^2 = 1$ is not a critical point and a noise stronger than the value Eq. (7) predicted.is necessary. Interestingly, the role of the 2nd moment changes at $\sigma^2 = 2$. It helps instability for $\sigma^2 > 2$. However, the forth moment term is always negative and against instability. When μ_4 grows faster than μ_2 with increasing σ^2, the L. H. S. is eventually becomes negative due to the dominance of the forth order fluctuation, which causes the reentrant transition. Since μ_4 enter Eq. (12) via $f^{(4)}$, the reentrant transition requires the presence of at least $-x^5$ in $f(x)$.

Figure 1 shows the order parameter $\langle x \rangle^*$ as a function of σ^2 obtained from a numerical simulation. The ordered phase appears for $1.3 < \sigma^2 < 5.5$. In Fig. 2, the growth of moments predicted by a numerical simulation is shown. The fourth moment indeed grows much faster than the second moment.

The type of the bifurcation is determined by the coefficient to ϵ^3 in Eq. (11). Depending on the magnitude of μ_2^* the bifurcation can be either supercritical or subcritical. A subcritical bifurcation can be realized by using a certain form of $f(x)$ or $g(x)$, which leads to a first order phase transition.[8, 9]

Stability along other directions can be phenomenologically determined. Since we assume that the probability distribution is unique for a given $\langle x \rangle$, there is only one fixed point for a given $\langle x \rangle^*$. If a moment is displaced from

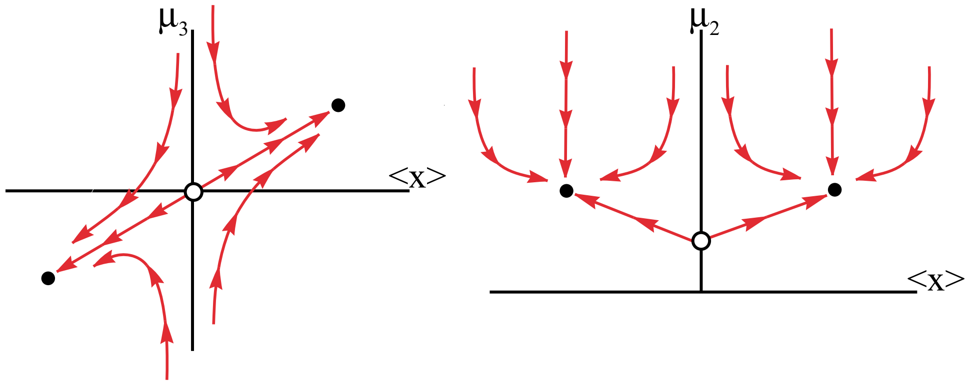

Figure 3. Schematic phase portraits projected on $\langle x \rangle$-μ_3 plane (left) and on the $\langle x \rangle -$ \$$\mu_2$ plane (right). There are one unstable fixed point (open circle) and two stable fixed points (solid circles). Projection onto other planes including $\langle x \rangle$ are all similar. Note $\mu_2 \geq 0$ in the left diagram.

the fixed point in a direction perpendicular to $\langle x \rangle$, the moment comes back to its fixed point because there is no other attractor. Therefore, the component of flow vectors on μ_n axises always points toward the fixed point. Figure 2 schematically illustrates the phase portrait in the ordered phase. Since the odd order moments undergo a pitchfork bifurcation at the same critical point as $\langle x \rangle$, the unstable eigenvector lies in the subspace spanned by μ_{2n+1} and $\langle x \rangle$. However, there is only one unstable eigenvector and a macroscopic observer sees its projection to $\langle x \rangle$.

2.3. Gaussian Approximation

For more quantitative analysis we need to find μ_2^* and μ_4^* as functions of σ^2 and D. However, due to the non-Gaussian nature of the system, the determination of these moments requires the moments of all orders. Approximate solutions are possible when cumulants above a certain order are negligible. In the present work, we assumes the simplest case where the cumulants above the second order are all negligible (Gaussian approximation). Although this approximation is quantitatively not justified, it correctly predicts main features of the pure noise-induced phase transition.

Under the Gaussian approximation, one need to know only μ_2^*. From Eq. (10) and $\langle x \rangle^* = \mu_{2n+1}^* = 0$, the stationary solution at the critical point must satisfy

$$\sigma_c^2 + 2[2\sigma_c^2 - (1+D)]\mu_2^* + \left(3\sigma_c^2 - 4\right)\mu_4^* - 2\mu_6^* = 0 \tag{13}$$

Applying the properties of Gaussian distribution, $\mu_4 = 3\mu_2^2$ and $\mu_6 = 15\mu_2^3$. to Eqs. (12) and (13) we obtain

$$(\sigma_c^2 - 1) + 3(\sigma_c^2 - 2)\mu_2^* - 15(\mu_2^*)^2 = 0 \quad . \tag{14}$$

and

$$\sigma_c^2 + 2[2\sigma_c^2 - (1+D)]\mu_2 + 3\left(3\sigma_c^2 - 4\right)(\mu_2^*)^2 - 30(\mu_2^*)^3 = 0 \tag{15}$$

In summary, μ_2^* must satisfy stability condition for $\langle x \rangle$, Eq (14) and the stationarity condition Eq (15) simultaneously. That is possible only when σ^2 and D satisfy a certain relation which determines the phase boundary. Figure 4 illustrates the phase boundary obtained by the Gaussian approximation and the exact one. Qualitatively, they are in good agreement.

Since what we needed was relation between μ_2^* and higher moments, we can assume other types of probability distribution such as $P(x) \sim e^{-x^4}$ which is more reasonable near the equilibrium phase transition point. Then, the relations are

$$\mu_4^* = \frac{1}{2}\frac{\pi^2}{\Gamma^4(3/4)}(\mu_2^*)^2, \qquad \mu_6^* = \frac{3}{2}\frac{\pi^2}{\Gamma^4(3/4)}(\mu_2^*)^3. \tag{16}$$

which yield nearly the same result as the Gaussian approximation.

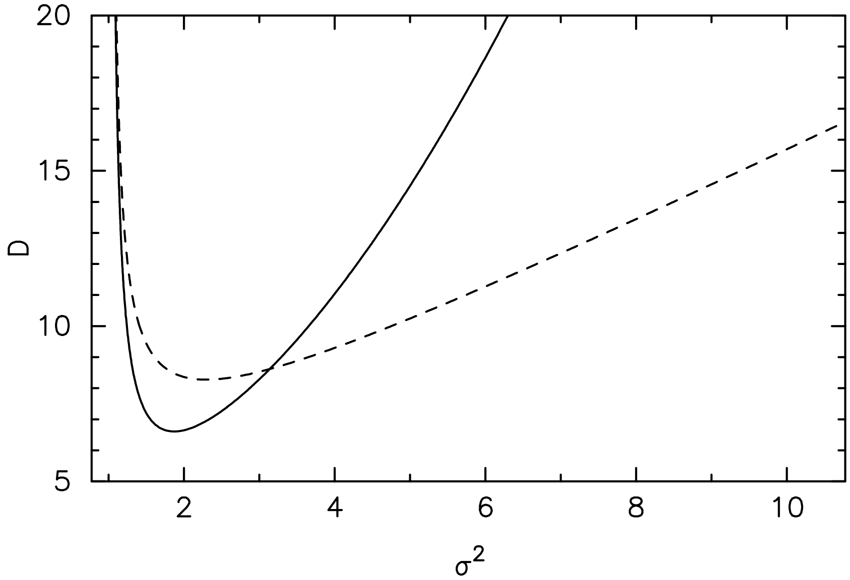

Figure 4. Phase boundary of the noise induced order. The dashed line is an exact solution and the solid line is based on the Gaussian approximation.

3. NOISE-INDUCED LIMIT CYCLE

We now consider a system with two degrees of freedom at each site. In order to keep a clear connection to the pure noise-induced phase transition discussed in the previous section, the following pair of stochastic differential equations are considered:

$$\dot{x_i} = f(x_i) - \frac{D}{N}\sum_j (x_i - x_j) + g(x_i)\xi_i(t) - y_i \tag{17}$$

$$\dot{y_i} = kx_i \tag{18}$$

where $f(x)$ and $g(x)$ are defined by Eq. (3) as before. When $y_i = 0$, Eq. (17) is identical to Eq. (1) and a bistable state is formed via noise-induced phase transition. One can imagine that once $\langle x \rangle$ becomes nonzero, $\langle y \rangle$ grows through Eq. (18). In turn, $\langle y \rangle$ pulls $\langle x \rangle$ to the opposite side of the bistable fixed points via Eq. (17). Therefore, $\langle x \rangle$ and $\langle y \rangle$ oscillate in time, which is reminiscent to a deterministic relaxation oscillation.

Using the mean field method used in the previous section, the dynamics of the mean values are expressed in the moment expansion:

$$\begin{aligned} \langle \dot{x} \rangle &= (\sigma^2 - 2)\mu_{3,0} - \mu_{5,0} + \left[\sigma^2 - 1 + 3(\sigma^2 - 2)\mu_{2,0} - 5\mu_{4,0}\right]\langle x \rangle \\ &\quad -10\mu_{3,0}\langle x \rangle^2 + (\sigma^2 - 2 - 10\mu_{2,0})\langle x \rangle^3 - \langle x \rangle^5 - \langle y \rangle \end{aligned} \tag{19}$$

$$\langle \dot{y} \rangle = k\langle x \rangle \tag{20}$$

where $\mu_{n,m} = \langle (x - \langle x \rangle)^n (y - \langle y \rangle)^m \rangle$. The dynamics of the moments are determined by simultaneous differential equations:

$$\begin{aligned} \dot{\mu}_{n,m} &= -nD\mu_{n,m} - n\mu_{n-1,m+1} + mk\mu_{n+1,m-1} + mk\mu_{n,m-1}\langle x \rangle - n\mu_{n-1,m}\langle y \rangle \\ &+ \sum_{\ell=0} \frac{n\mu_{n+\ell-1,m}}{\ell!}\left\{f^{(\ell)}(\langle x \rangle) + \frac{\sigma^2}{2}[g'(\langle x \rangle)g(\langle x \rangle)]^{(\ell)}\right\} + \sum_{\ell=0} \frac{n(n-1)\mu_{n+\ell-2,m}}{\ell!}\frac{\sigma^2}{2}[g^2(\langle x \rangle)]^{(\ell)} \end{aligned} \tag{21}$$

where $\mu_{0,0} = 1$ and $\mu_{1,0} = \mu_{0,1} = 0$.

When the noise intensity is small, the system has a stable fixed point at $\langle x\rangle^* = 0$ and $\langle y\rangle^* = (\sigma^2-2)\mu^*_{3,0} - \mu^*_{5,0}$ Even order moments are not necessarily zero but odd order moments $\mu^*_{2n+1,0}$ must be zero due to the symmetry of the system, implying $\langle y\rangle^* = 0$. Then, other odd order moments, $\mu^*_{2n+1,2m}$ and $\mu^*_{2n,2m+1}$ are also zero.

The number of degrees of freedom is doubled from the case of the macroscopic pitchfork bifurcation. However, the basic idea used for the pitchfork bifurcation in the previous section can be naturally extended to the present case. Since the stationary probability distribution is unique for a given $\langle x\rangle^*$ and $\langle y\rangle^*$, any instability must involve the $\langle x\rangle$-$\langle y\rangle$ plane. In another word, the unstable subspace is not perpendicular to the $\langle x\rangle$-$\langle y\rangle$ plane. Therefore, it is sufficient to investigate the stability of the fixed point on the $\langle x\rangle$-$\langle y\rangle$ plane by fixing all moments at the fixed point. The dynamics of small displacements $\epsilon_x = \langle x\rangle - \langle x\rangle^*$ and $\epsilon_y = \langle y\rangle - \langle y\rangle^*$ constrained on the plane is determined by

$$\begin{aligned}\dot{\epsilon}_x &= \left[\sigma^2 - 1 + 3(\sigma^2-2)\mu^*_{2,0} - 5\mu^*_{4,0}\right]\epsilon_x \\ &\quad +(\sigma^2 - 2 - 10\mu^*_{2,0})\epsilon_x^3 - \epsilon_x^5 - \epsilon_y \qquad (22)\\ \dot{\epsilon}_y &= k\epsilon_x \qquad (23)\end{aligned}$$

A standard linear stability analysis finds that the fix point becomes unstable at a critical point.

$$\sigma_c^2 - 1 + 3(\sigma_c^2 - 2)\mu^*_{2,0} - 5\mu^*_{4,0} = 0 \qquad (24)$$

which is identical to Eq. (12). This is a Hopf bifurcation leading to a limit cycle state. Once $\langle x\rangle$ and $\langle y\rangle$ start oscillating, moments are all time-varying.. However, the dimension of unstable subspace is two and the actual attractor is still a limit cycle.

In order to find a critical noise intensity, we must find $\mu^*_{2,0}$ and $\mu^*_{4,0}$. From Eq. (21), the stationary condition for $\mu_{2,0}$ is

$$\sigma_c^2 + 2[2\sigma_c^2 - (1+D)]\mu^*_{2,0} + \left(3\sigma_c^2 - 4\right)\mu^*_{4,0} - 2\mu^*_{6,0} = 0 \qquad (25)$$

where $\mu^*_{1,1} = 0$ is assumed from the symmetry consideration. Equations (24) and (25) are identical to the Eqs. (13) and (14), respectively. Therefore, within the Gaussian approximation, the two cases share the same phase boundary. Exactly speaking, the moments are different between two cases due to the cross correlation between x and y. However, such an effect is small and one can expect that the two cases approximately share the same critical points.

In Fig. 1 1 an order parameter $r = \sqrt{\langle\langle(\langle x\rangle^2 + \langle y\rangle^2)\rangle\rangle}$ for the Hopf bifurcation as well as $\langle x\rangle$ for the pitchfork bifurcation are plotted. ($\langle\langle\cdots\rangle\rangle$ indicates temporal average.) The order phase appears in the same parameter region for both cases. The oscillatory motion of $\langle x\rangle$ illustrated in Fig. 5 shows a typical pattern of relaxation oscillation. The power spectrum shown in Fig. 5 indicates it is a periodic without much of background noise.

Fig 6 shows snapshots of the distribution of the individual systems around the mean value in the ordered state over one oscillation period. Even for this rather small number of systems (N=625) one can see a quite regular motion of the mean while the individual systems are spread widely around the mean.

4. CONCLUSIONS AND DISCUSSIONS

Unlike a linear stochastic system, time-evolution of macroscopic quantities depends on all moments. Therefore, the deterministic dynamics of macroscopic quantities coupled to infinite set of simultaneous ordinary differential equations. However, when such nonlinear systems undergo a symmetry breaking phase transition, the dimension of unstable subspace is the same as the dimension of macroscopic quantities. We showed that without solving eigenvalues of infinitely large degrees of freedom, one can investigate the bifurcation in macroscopic dynamics.. This method also allow us to use standard bifurcation theories to investigate the noise-induced phase transition. Furthermore, when quantitative analysis is necessary, the present methods allow us to construct an approximate dynamical system consisting of several degrees of freedom if higher order cumulants are negligible.

Using this method, we investigated a macroscopic pitchfork bifurcation and a Hopf bifurcation induced by multiplicative noise. In the latter case, the macroscopic quantities oscillate in time when the system is in an

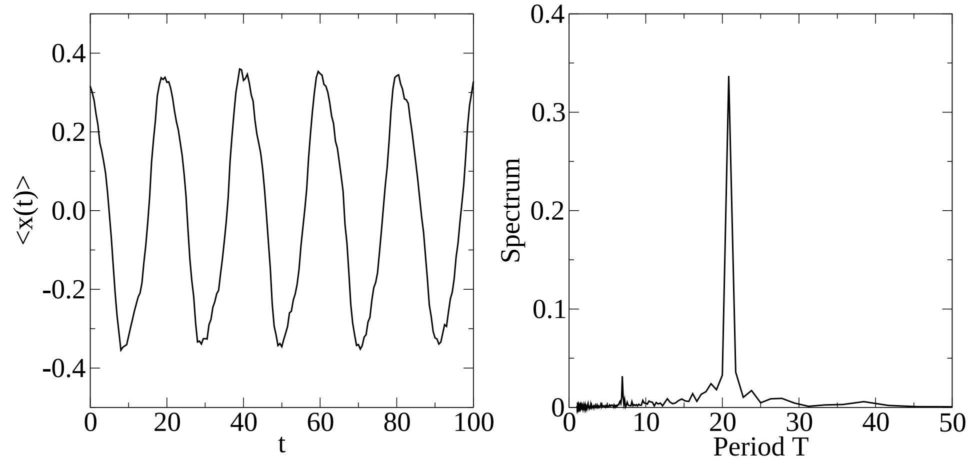

Figure 5. A limit cycle trajectory of $\langle x \rangle$ (left panel) and its power spectrum as a function of the period of the oscillation (right panel). Parameter values are K=10, σ^2=2, and k=0.1.

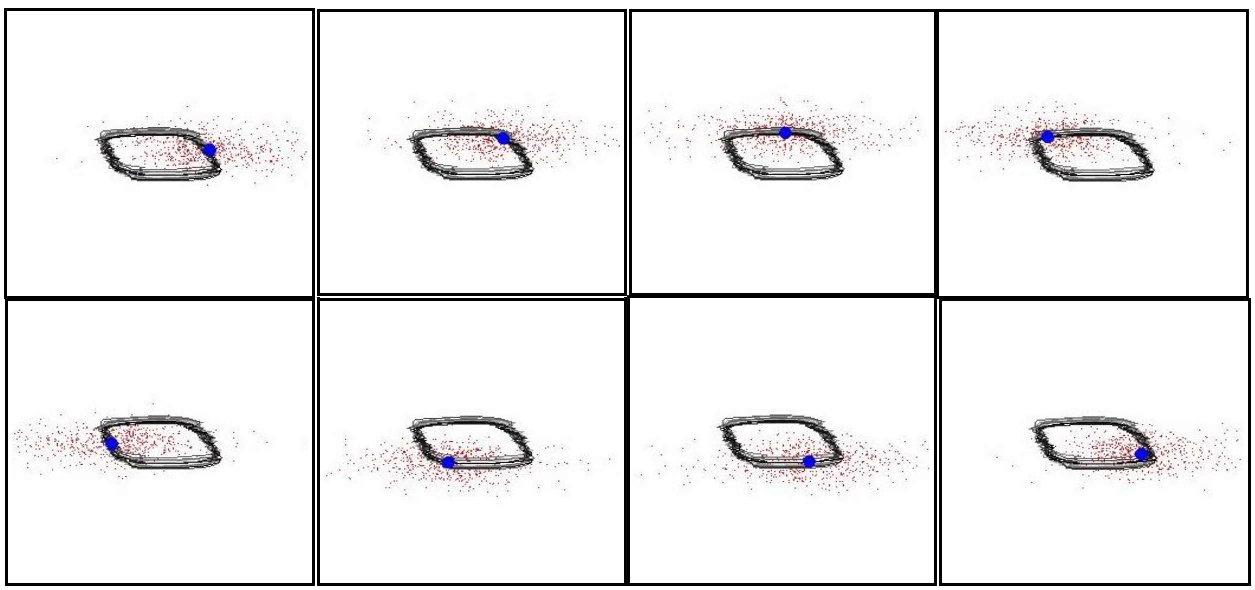

Figure 6. Sequence of snapshots of 625 coupled systems in parameter space. Time goes from left to right in the upper row and then from left to right in the lower row. Small dots symbolize the individual systems, the big dot symbolizes the mean value and the line shows the trajectory of the mean.

ordered phase. This oscillation is purely induced by noise via spontaneous symmetry breaking. The macroscopic oscillation suggests a strong synchronization of microscopic degrees of freedom despite the presence of noise. Actually, it is the noise that generates the macroscopic order. This phenomena is another example where noise positively contributes to the construction of order. Furthermore, internal fluctuation affect the formation of the ordered phase. When the fluctuation is too large, the synchronization is killed not by the second moment but by the fourth moment, leading to the reentrant transition.

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