Synchrony and Critical Behavior: Equilibrium Universality in Nonequilibrium Stochastic Oscillators

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Abstract. We review our work on a discrete model of stochastic, phase-coupled oscillators that is sufficiently simple to be characterized in complete detail, lending insight into the universal critical behavior of the corresponding nonequilibrium phase transition to macroscopic synchrony. In the mean-field limit, the model exhibits a supercritical Hopf bifurcation and global oscillatory behavior as coupling eclipses a critical value. The simplicity of our model allows us to perform the first detailed characterization of stochastic phase coupled oscillators in the locally coupled regime, where the model undergoes a continuous phase transition which remarkably displays signatures of the XY equilibrium universality class, verifying recent analytical predictions. Finally, we examine the effects of spatial disorder and provide analytical and numerical evidence that such disorder does not destroy the capacity for synchronization.

Keywords: stochastic phase coupled oscillators, synchronization, critical behavior, universality
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Time and periodicity play a critical role in a host of physical, biological, and chemical systems [1, 2]. In particular, a vast number of systems consist of noisy individual entities characterized by periodic (oscillatory) dynamics that give rise to a potentially complex competition between individual dynamics on a microscopic scale and large scale cooperative behavior. Originating with the pioneering work of Kuramoto [3, 4], the scientific literature is replete with studies of such synchronization [1, 2]. The typical approach involves systems of coupled, nonlinear differential equations. As such, work has been traditionally limited to relatively small, deterministic systems in the mean field limit [5, 6, 7, 8, 9].

While providing a mature understanding of the synchronization dynamics in coupled oscillators, the numerical and analytical complexities of these familiar models have prohibited a complete characterization in the large system limit. Therefore, the analogy between the cooperative behavior of these oscillators with phase transitions and critical phenomena from statistical mechanics [10, 11] has not been fully exploited. Specifically, the onset of synchronization when viewed on a long wavelength scale is characterized by a macroscopic change in an inherently nonequilibrium symmetry: for systems below the synchronization threshold, the large-scale dynamics appears time-translationally invariant, while synchronized systems oscillate collectively, breaking this time-based symmetry.

With these possible analogies in mind, and with an eye towards critical behavior, we develop a simple, discrete model of noisy phase-coupled oscillators [12, 13, 14] which displays the desired macroscopic behavior while remaining sufficiently simple to be characterized as a phase transition. Owing to the well-established notion of universality, which renders microscopic specifics essentially irrelevant for study of the typically long-wavelength behavior occurring near criticality, our model of the phase transition in question must preserve the macroscopic symmetry-breaking while incorporating the relevant physical details, namely, stochasticity and microscopic periodicity.

Our starting point is a three-state unit [12, 13, 14] governed by transition rates g (Fig. 1). Loosely speaking, we interpret the state designation as a generalized (discrete) phase, and the transitions between states, which we construct to be unidirectional, as a phase change and thus an oscillation of sorts. The probability of going from the current state i to state $i+1$ in an infinitesimal time dt is gdt, with i=1,2,3 modulo 3. For an isolated unit, the transition rate is simply a constant (g) that sets the oscillator's intrinsic frequency; for many coupled units, we will allow the transition rate to depend on the neighboring units in the spatial grid, thereby coupling neighboring phases.

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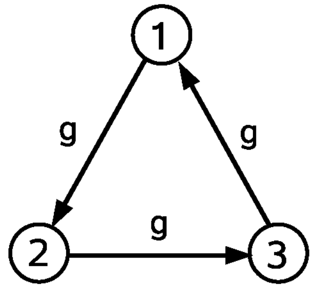


FIGURE 1. Three-state unit with generic transition rates g.

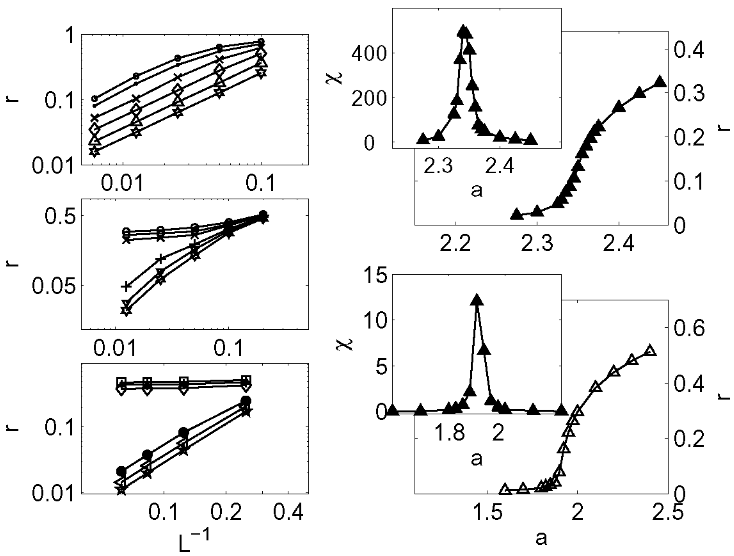


FIGURE 2. Dimensionality (d) dependent phase transition. Left column shows r vs L on a log-log scale for $d = 2, 3, 4$, top to bottom. For $d = 2$, the curves start from $a = 2,0$ (top) with increments of 0.5. For $d = 3$, the curves start from $a = 2.275$ (top) with increments of 0.025 (with the exception of $a = 2.350$, not shown). For the $d = 4$ case, the curves are $a = 1.6,1.7,1.8,2.0,2.1,2.2$, top to bottom. In the limit of infinite system size, a synchronous phase emerges for $d > 3$ for sufficiently large a. The right column shows r vs a and χ vs a for $d = 3$, $L = 80$ (top) and $d = 4$, $L = 16$, bottom. Fluctuations peak near the critical point, giving an estimation of $a_c = 2.345 \pm 0.005$ and $a_c = 1.900 \pm 0.025$ for $d = 3$ and $d = 4$, respectively.

For an isolated unit we write the linear evolution equation $\partial P(t)/\partial t = MP(t)$, where the components $P_i(t)$ of the column vector $P(t) = (P_1(t)\ P_2(t)\ P_3(t))^T$ are the probabilities of being in state i at time t, and

$$M = \begin{pmatrix} -g & 0 & g \\ g & -g & 0 \\ 0 & g & -g \end{pmatrix}. \tag{1}$$

The system reaches a steady state for $P_1^* = P_2^* = P_3^* = 1/3$. The transitions $i \to i+1$ (with $i+1 \equiv 1$ when $i = 3$) occur with a rough periodicity determined by g; that is, the time evolution of our simple model qualitatively resembles that of the discretized phase of a generic noisy oscillator.

We implement microscopic coupling by allowing the transition probability of a given unit to depend on the states of the unit's nearest neighbors in a spatial grid. Specifically, we choose a function which compares the phase at a given site with its neighbors, and adjusts the phase at the given site so as to facilitate phase coherence. We settle on the following form of the transition rate from state i to state j:

$$g_{ij} = g \exp\left[\frac{a(N_j - N_i)}{2d}\right] \delta_{j,i+1}. \tag{2}$$

Here the constant a is the coupling parameter, g is a parameter related to the intrinsic frequency of each oscillator, and δ is the Kronecker delta. N_k is the number of nearest neighbors in state k, and $2d$ is the total number of nearest neighbors in d dimensional cubic lattices. While this choice is by no means unique and these rates are somewhat distorted by their independence of the number of nearest neighbors in state $i-1$, the form (2) does lead to synchronization and allows for fast numerical simulation of large lattices near the critical regime [12, 13].

To verify the emergence of global synchrony, we first consider a mean field version of the model. In the large N limit with all-to-all coupling we write

$$g_{ij} = g \exp\left[a(P_j - P_i)\right] \delta_{j,i+1}. \tag{3}$$

Note that in the mean field limit g_{ij} does not depend on the location of the unit within the lattice. Also, there is an inherent assumption that we can replace N_k/N with P_k. With this simplification we arrive at a nonlinear equation for the mean field probability, $\partial P(t)/\partial t = M[P(t)]P(t)$, with

$$M[P(t)] = \begin{pmatrix} -g_{12} & 0 & g_{31} \\ g_{12} & -g_{23} & 0 \\ 0 & g_{23} & -g_{31} \end{pmatrix}. \tag{4}$$

Normalization allows us to eliminate P_3 and obtain a closed set of equations for P_1 and P_2. We can further characterize the mean field solutions by linearizing about the fixed point $(P_1^*, P_2^*) = (1/3, 1/3)$. The complex conjugate eigenvalues of the Jacobian evaluated at the fixed point, $\lambda_{\pm} = g(2a - 3 \pm i\sqrt{3})/2$, cross the imaginary axis at $a = 1.5$, indicative of a Hopf bifurcation at this value, which following a more detailed analysis [12, 13] can be shown to be supercritical. Hence, as a increases, the mean field undergoes a qualitative change from disorder to global oscillations, and the desired breaking of time translational symmetry emerges. Numerical solutions confirm this behavior, yielding results that agree with simulations of an all-to-all coupling array [12, 13].

We now proceed with a detailed numerical characterization of the locally coupled case. We performed simulations of the locally coupled model in continuous time on d-dimensional cubic lattices with periodic boundary conditions. Time steps were 10 to 100 times smaller than the fastest local average transition rate, i.e., $dt \ll e^{-a}$ (we set g=1). We find that much smaller time steps lead to essentially the same results. Starting from random initial conditions, all simulations were run until an apparent steady state was reached, and statistics are based on 100 independent trials.

Following other works on phase synchronization, we introduce the order parameter [3, 4]

$$r = \langle R \rangle, \quad R \equiv \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\phi_j} \right|. \tag{5}$$

Here ϕ_j is the discrete phase $2\pi(k\text{-}1)/3$ for state $k \in \{1,2,3\}$ at site j. The brackets represent an average over time in the steady state and over all independent trials. Nonzero r in the thermodynamic limit indicates synchrony. We also calculate the generalized susceptibility $\chi = L^d[\langle R^2 \rangle - \langle R \rangle^2]$.

As shown in Fig. 2, the model undergoes a dimension-dependent transition marked by characteristics of a typical phase transition, including a macroscopic change in r in the infinite system limit, a peak in fluctuations at the critical point, and a diverging spatial correlation length [12, 13] (not shown). Specifically, in $d = 2$ we do not see the emergence of a synchronous phase. Instead, we observe intermittent oscillations (for very large values of a) that decrease drastically with increasing system size. In fact, $r \to 0$ in the thermodynamic limit, even for very large values of a. We conclude that the phase transition to synchrony cannot occur for $d = 2$.

In contrast to the d=2 case, which serves as the lower critical dimension, a clear thermodynamic-like phase transition occurs in three dimensions. Figure 2 shows explicitly that for $a < a_c$ when $d = 3$, $r \to 0$ as system size is increased, and a disordered phase persists in the thermodynamic limit. For $a > a_c$, the order parameter approaches a finite value as the system size increases. In addition, Fig. 2 shows the behavior of the order parameter as a is increased for the largest system studied (L=80); the upper left inset shows the peak in χ at $a = 2.345 \pm 0.005$, thus providing an estimate of the critical point a_c. While the accuracy of our current estimation of the critical point is modest, it nonetheless suffices to determine the universality class of the transition. Similarly, for $d = 4$ we estimate the transition coupling to be $a_c = 1.900 \pm 0.025$ (Fig. 2), and for $d = 5$ we see a transition to synchrony at $a_c = 1.750 \pm 0.015$ [12, 13] (not shown).

To further characterize this transition, we use finite size scaling analysis by assuming the standard scaling

$$r = L^{-\frac{\beta}{\nu}} F[(a - a_c)L^{\frac{1}{\nu}}]. \tag{6}$$

Here $F(x)$ is a scaling function that approaches a constant as $x \to 0$. To test our numerical data against different universality classes we choose the appropriate critical exponents for each, recognizing that there are variations in the

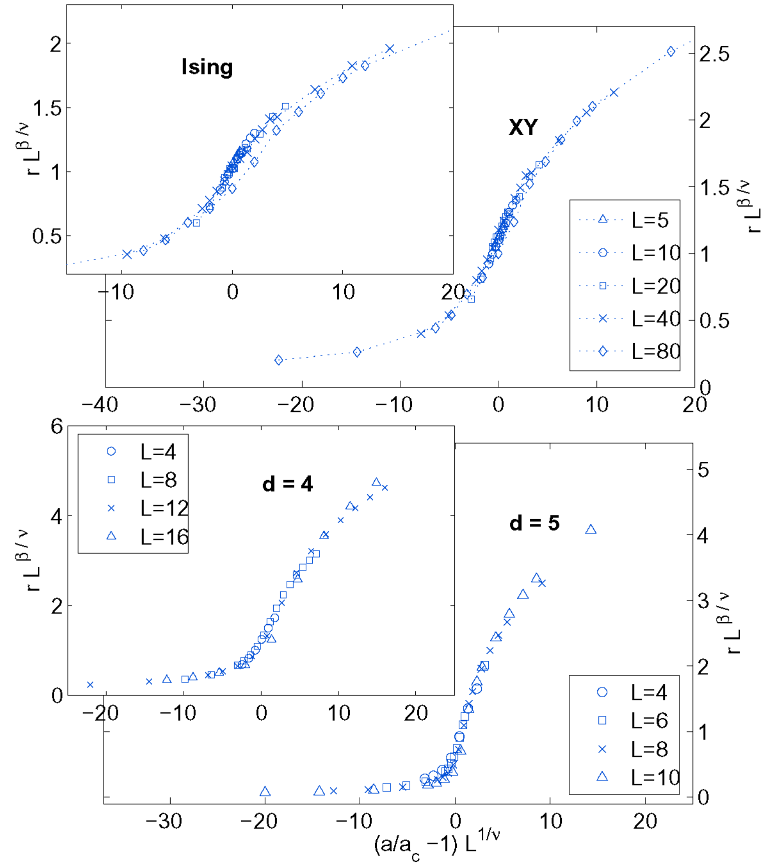


FIGURE 3. Finite size scaling analysis for $d = 3$, $d = 4$, $d = 5$: Data collapse using ansatz (6) with mean field exponents.

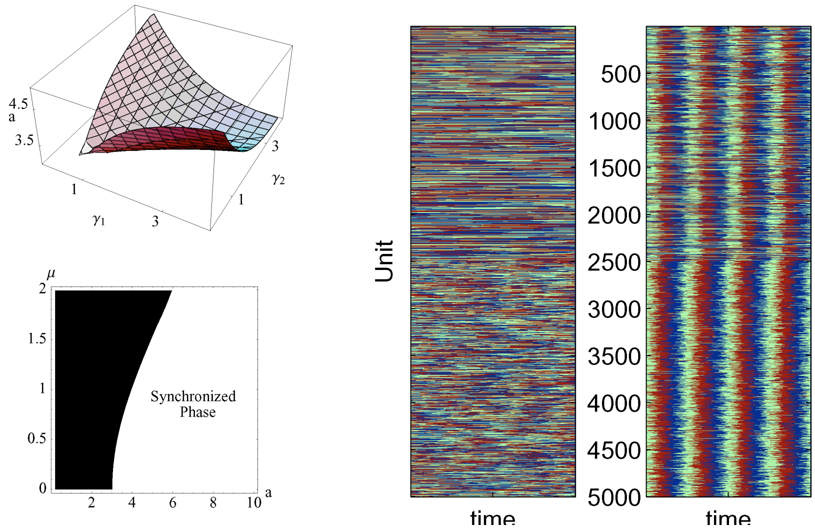
reported values of these exponents [15]. For the XY universality class we use the exponents β=0.34 and ν=0.66 [17]. For the Ising exponents we use β=0.31 and ν=0.64 [11]. In Fig. 3, we see quite convincingly a collapse when exponents from the XY class are used, verifying recent analytic predictions [18, 19]. For comparison, we also show the data collapse with $d = 3$ Ising exponents (note the scale differences). Because we expect $d = 4$ to be the upper critical dimension in accordance with XY/Ising behavior, we anticipate a slight breakdown of the scaling relation (6). Nevertheless, as shown in Fig. 4, the data collapse is very good with the mean field exponents. As such, our simulations suggest that $d = 4$ serves as the upper critical dimension. The case $d = 5$, where the data collapse with the mean field exponents is excellent, is also shown in Fig. 3, and further supports the claim that d_{uc}=4.

Having observed convincingly equilibrium-like critical behavior in systems of inherently nonequilibrium oscillators, we now turn to the question of spatial disorder. That is, when units are no longer identical (g not equal for all units), is the capacity for synchronization destroyed? To address this question, we consider first a dichotomously disordered population consisting of two subpopulations of oscillators, each characterized by a different frequency parameter $g = \gamma_1$ and $g = \gamma_2$. We assume a modified form of the inter-unit coupling given by

$$g_{ij} = g \exp\left[\frac{a(N_j - N_{i-1})}{n}\right] \delta_{j,i+1}, \tag{7}$$

where n is the number of oscillators to which unit ν is coupled, and N_k is the number of units among the n that are in state k. This form maintains a closer macroscopic analogy with coupled oscillators far above the frequency threshold [16], and while more time consuming for in depth studies of the critical regime, it proves more suitable for the current disorder studies.

In the mean field limit, our dichotomously disordered array can be characterized by a six dimensional equation for P_{i,γ_1} and P_{i,γ_2}, $i \in 1,2,3$, where, for example, P_{1,γ_1} represents the probability for a unit with transition rate parameter



(a) **(b)**

FIGURE 4. (**a.**) Upper panel: The critical surface a_c is shown for a dichotomously disordered system. Specifically, the boundary given by the contour $\mathrm{Re}\lambda_+ = 0$ is plotted in (γ_1, γ_2, a) space. This contour indicates the critical point, where the Hopf bifurcation occurs and the disordered solution becomes unstable. The region above the contour represents the synchronized phase. Lower panel: Stability boundary in terms of relative width parameter. (**b.**) Time evolution snapshots of a dichotomously disordered system with $\gamma_1 = 0.5$ and $\gamma_2 = 1.5$ are shown for $a = 3.5 < a_c$ (left) and $a = 4.1 > a_c$ (right). Each color represents the state of the oscillator.

γ_1 to be in state 1. Following normalization, the set reduces to four equations which can be linearized about the nonsynchronous fixed point $(1/3, 1/3, 1/3, 1/3)$. The eigenvalues of the relevant Jacobian are given by:

$$\begin{aligned} \frac{\mathrm{Re}\lambda_\pm}{\gamma_1+\gamma_2} &= \frac{1}{8}\left[a-6 \pm B(a,\mu)\cos\left(C(a,\mu)\right)\right], \\ \frac{\mathrm{Im}\lambda_\pm}{\gamma_1+\gamma_2} &= \frac{1}{8}\left[\sqrt{3}(a+2) \pm B(a,\mu)\sin\left(C(a,\mu)\right)\right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} B(a,\mu) &\equiv \sqrt{2}\left[a^4 - 6a^2\mu^2 + 3\mu^4(a^2+3)\right]^{1/4}, \\ C(a,\mu) &\equiv \frac{1}{2}\tan^{-1}\left(\frac{-\sqrt{3}(a^2-(a+3)\mu^2)}{a^2+3(a-1)\mu^2}\right). \end{aligned} \tag{9}$$

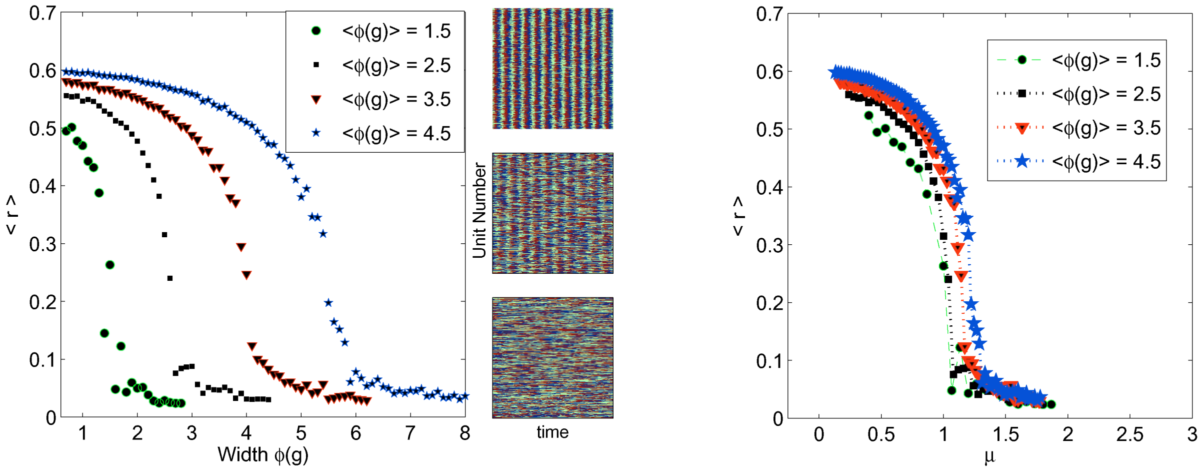
In fact, one pair of eigenvalues crosses the imaginary axis (Hopf bifurcation) at a critical value $a = a_c$, but the other pair shows no qualitative change as a is varied. Aside from an overall factor $(\gamma_1 + \gamma_2)$, Eqs. (8) depend only on the relative width variable

$$\mu \equiv \frac{2(\gamma_1 - \gamma_2)}{(\gamma_1 + \gamma_2)} \tag{10}$$

$(-2 \le \mu \le 2)$, and in fact the Hopf bifurcation occurs at a single value of $a_c(\mu)$. As shown in Fig. 4, the surface a_c separates regions of synchronous and asynchronous phases. A small-μ expansion leads to an estimate of a_c to $O(\mu^2)$,

$$a_c \approx \frac{1}{8}\left(12 + 3\mu^2 + \sqrt{3}\sqrt{(12+\mu^2)(4+3\mu^2)}\right), \tag{11}$$

a result that exhibits these trends explicitly. In particular, we note the increase in a_c with increasing μ. Figure 4 also shows snapshots from simulations of a globally coupled, dichotomously disordered population ($N = 5000$ units, or 2500 with $\gamma_1 = 0.5$ and 2500 with $\gamma_2 = 1.5$), confirming the emergence of macroscopic synchrony for sufficiently strong coupling.



(a) **(b)**

FIGURE 5. (**a.**) As the width of the $\phi(g)$ distribution increases, a critical width is reached beyond which synchronization is destroyed. The coupling is chosen to be $a = 3.2$, and the four curves represent the steady state, time-averaged order parameter for distributions with different means. As the mean of the $\phi(g)$ distribution increases, the transition to disorder occurs at a greater width. The insets at the right show the long-time behavior of an entire population of mean transition rate parameter 3.5 (corresponding to the triangle order parameter data) and widths of 0.6, 4.0, and 6.2. (**b.**) The same data is plotted against μ, the relative width parameter.

Finally, we show that these trends carry over to fully disordered systems, where γ_i is drawn from a uniform distribution characterized by a relative width variable μ. Explicitly, μ is the ratio of the distribution mean to the distribution width. As shown in Fig. 5, synchronization still occurs in these systems and furthermore, the critical coupling a_c is essentially determined by the distribution parameter μ. We find that, analogous to the dichotomously disordered system, synchronization occurs at a *single* value of a_c which depends crucially on μ.

In conclusion, we have demonstrated the remarkable result that a fundamentally nonequilibrium transition, namely, a phase transition that breaks the symmetry of translation in time, is described by an equilibrium universality class. By utilizing a simple discrete model for active noisy oscillators, we have shown compelling numerical and analytical evidence that the emergence of synchronous oscillations in these systems contains signatures of an equilibrium phase transition, including diverging fluctuations at criticality, a macroscopic change in the order parameter, and classic exponents belonging to the XY universality class. In addition, we show that synchronization can and does occur in both dichotomously and fully disordered populations, leading to large-scale cooperativity in spite of the non-identical nature of the constituents.

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