Finite time thermodynamics of a quantum dot heat engine

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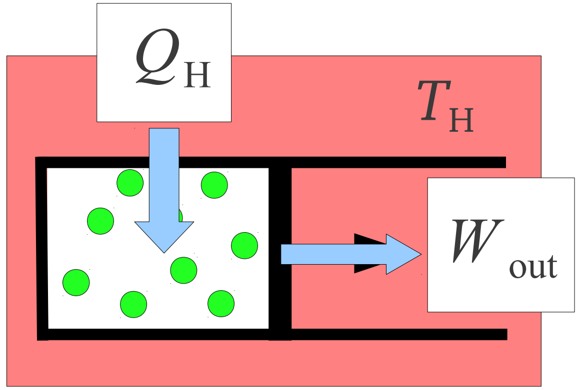
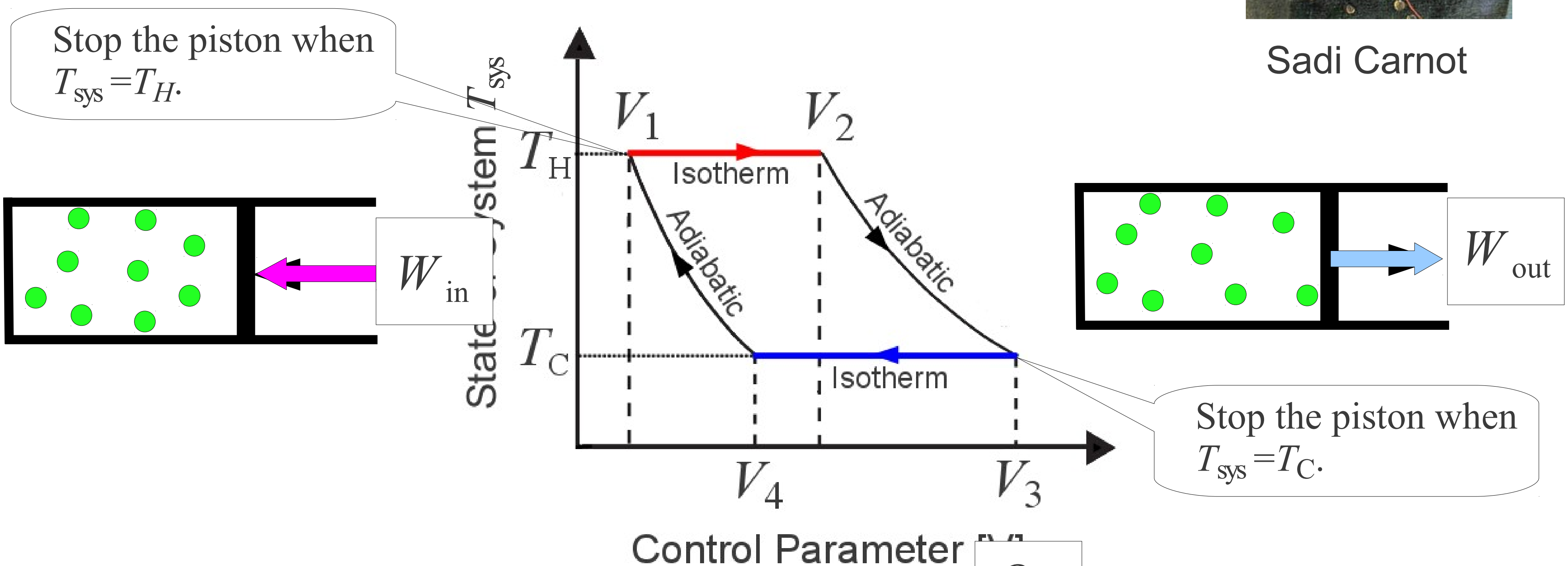
Carnot Cycle

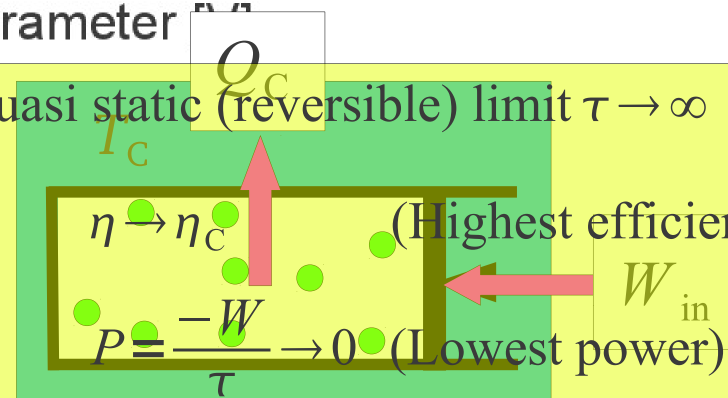


Sadi Carnot



1st Law: $W_{Net} + Q_H + Q_C = 0$

2nd law: $\eta = \frac{-W_{Net}}{Q_H} = \le 1 - \frac{T_C}{T_H} \equiv \eta_C$



Quasi static (reversible) limit $\tau \to \infty$

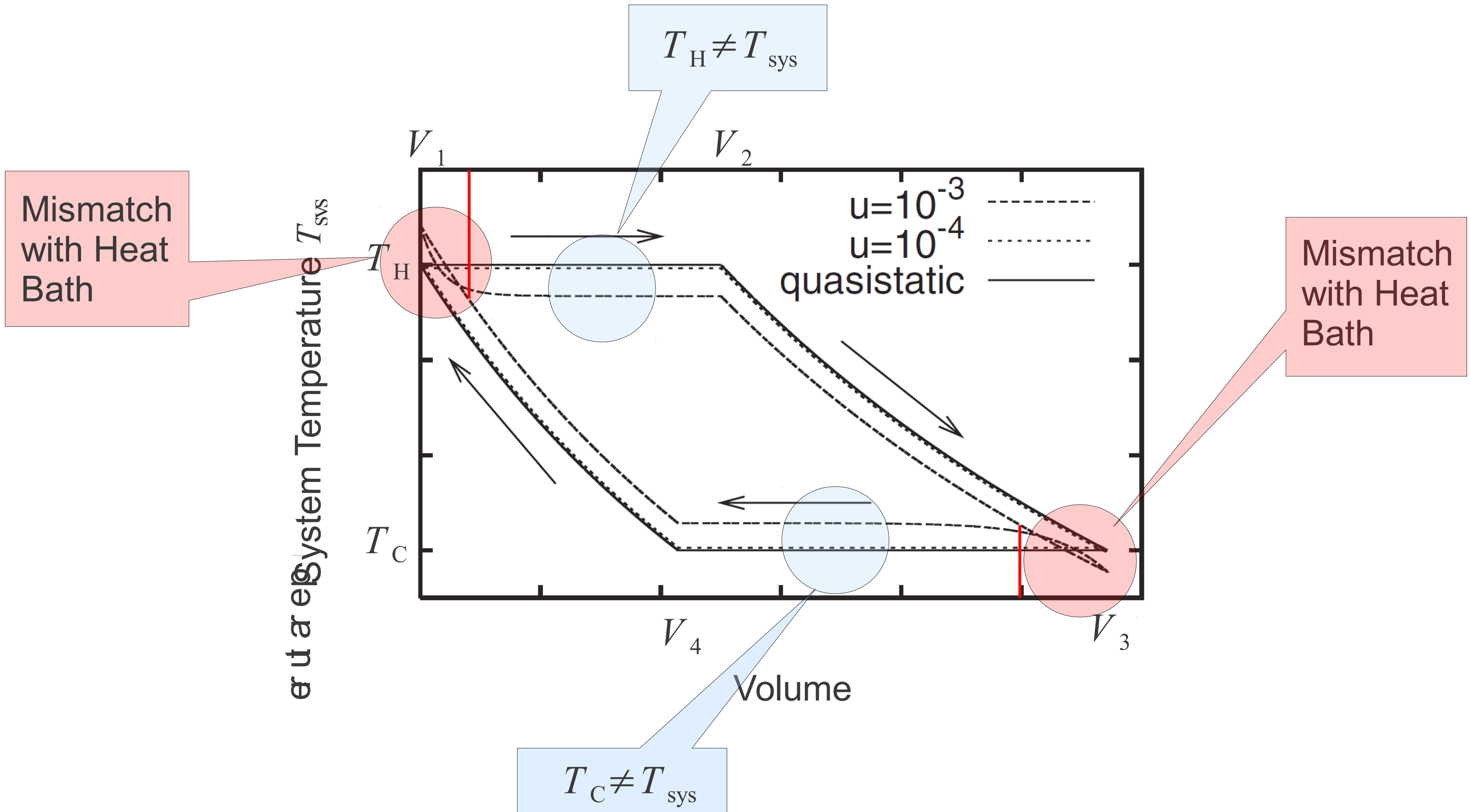
$\eta \to \eta_C$ (Highest efficiency)

$P = \frac{-W}{\tau} \to 0$ (Lowest power)

Finite Time Thermodynamics

Question: What will happen if τ is finite?

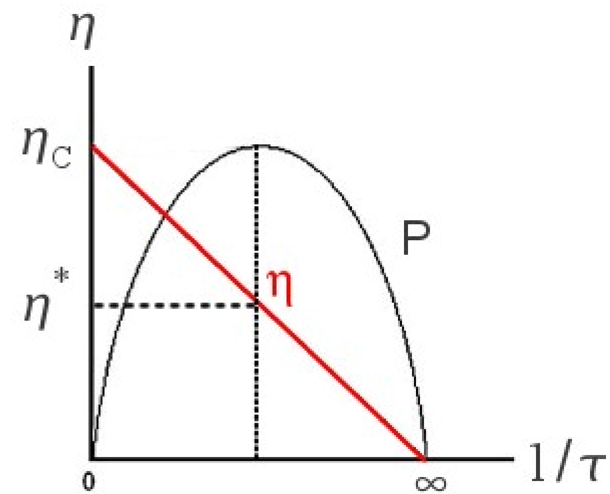
Textbook: The efficiency is lowered due to dissipation (entropy production).



Izumida and Okuda, EPL 83 (2008), 60003

Efficiency at maximum power

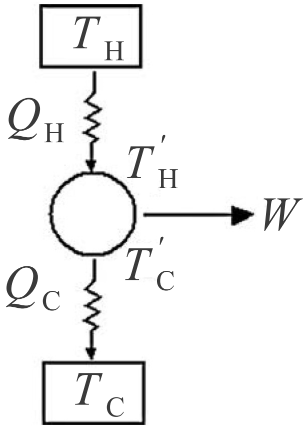
Efficiency decreases as τ increases.
But Power reaches its maximum at a certain τ.



maximum power: $$P^* = \frac{-W}{\tau} = \frac{Q_H(\tau_H^*) + Q_C(\tau_C^*)}{\tau_H^* + \tau_C^*}$$

efficiency at maximum power: $$\eta^* = 1 + \frac{Q_C(\tau_C^*)}{Q_H(\tau_H^*)}$$

Curzon-Ahlborn efficiency



Efficiency at maximum power

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}}$$

Novikov, J. Nucl. Energy **II** (1958), 125

Curzon-Ahlborn, Am. J. Phys **43** (1975), 22

Table I. Observed performance of real heat engines.

Power source	T_C	T_H	η_C	η_{CA}	$\eta_{Observed}$
West Thurrock (U.K.)[2] Coal Fired Steam Plant	~25	565	64.1%	40%	36%
CANDU (Canada)[4] PHW Nuclear Reactor	~25	300	48.0	28%	30%
Larderello (Italy)[5] Geothermal Steam Plant	80	250	32.3%	17.5%	16%
Steam power plant (USA)	298	923	67.6%	43.2%	40%

How universal is the Curzon-Ahlbone efficiency?

Many case studies:

$$\eta^* \approx \eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{16} + \cdots$$

Brownian heat engine [Schmiedl&Seifert, EPL **81** (2008), 20003]

$$\eta^* = \frac{2(T_H - T_C)}{3T_H + T_C} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{32} + \ldots$$

Linear non-equilibrium thermodynamics

$$\eta^* = \frac{\eta_C}{2}$$

Van den Broeck, PRL **95** (2005), 190602

Non-linear correction (with left-right symmetry)

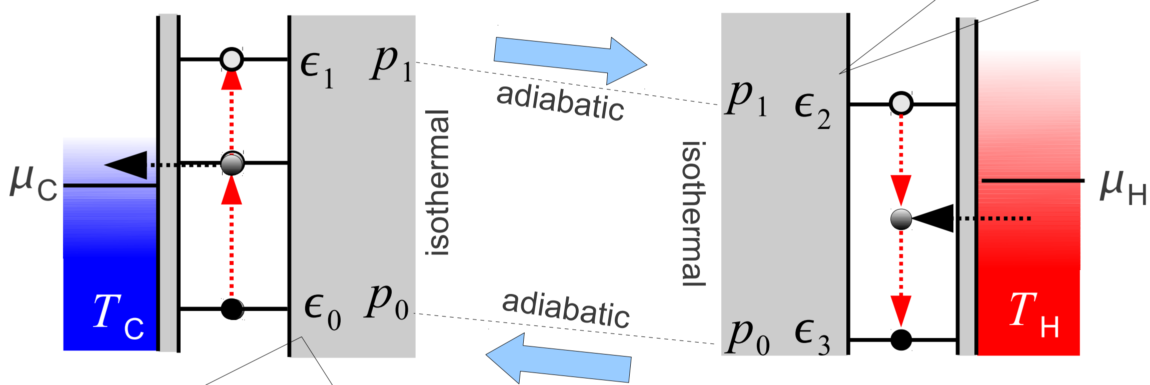
$$\eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + o(\eta_C^3)$$

Esposito *et al.*, PRL **102** (2009), 130602

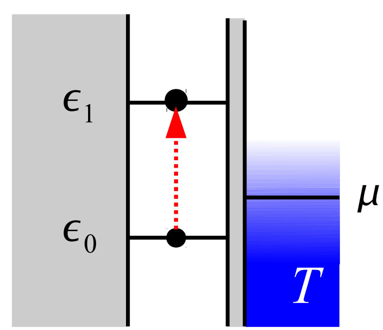
Quantum Dot Carnot Engine

$$p_1 = \frac{1}{e^{(\epsilon_1-\mu_H)/kT_H}+1} = \frac{1}{e^{(\epsilon_2-\mu_C)/kT_C}+1}$$

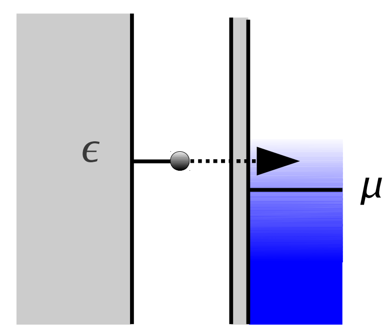
Control parameter (protocol) $\epsilon(t)$



$$p_0 = \frac{1}{e^{(\epsilon_0-\mu_H)/kT_H}+1} = \frac{1}{e^{(\epsilon_3-\mu_C)/kT_C}+1}$$



Work done to the system
$W = \epsilon_1 - \epsilon_0$



Heat flows out
$Q = \epsilon - \mu$

Esposito et al., PRE**81** (2010), 041106

Steps to find efficiency at maximum power

Step 1: Find $\varepsilon(t)$ that maximizes heat.

Step 2: Maximize power with respect to τ_H and τ_C

Step 3: Evaluate efficiency with optimum Q and τ

$$P = \frac{Q_H + Q_C}{\tau_H + \tau_C}$$

$$\eta = 1 + \frac{Q_C}{Q_H}$$

Master equation $\dot{p}(t) = -\omega_1 p(t) + \omega_2 [1 - p(t)] \Longrightarrow \dot{p}(t) = -C\, p(t) + \frac{C}{e^{\epsilon(t)/kT} + 1}$

$$\omega_1 = \frac{C}{e^{-\epsilon(t)/kT} + 1}, \quad \omega_2 = \frac{C}{e^{+\epsilon(t)/kT} + 1}$$

Work: $W[p(t)] = \int_0^\tau \dot{\epsilon}(t)\, p(t)\, dt$

Heat: $Q[p(t)] = \int_0^\tau \epsilon(t)\, \dot{p}(t)\, dt$

Step 1: Maximizing heat

$$Q[p(t)] = \int_0^{\tau} \epsilon(t)\,\dot{p}(t)\,dt = \int_0^{\tau} \ln\left[\frac{1}{C\,p(t)+\dot{p}(t)} - 1\right]\dot{p}(t)\,dt \equiv \int_0^{\tau} L(p,\dot{p})\,dt$$

$$L(p,\dot{p}) = \ln\left[\frac{C}{C\,p(t)+\dot{p}(t)} - 1\right]\dot{p}(t)$$

$$\delta \int L\,dt = 0 \quad \Rightarrow \quad L - \dot{p}\frac{\partial L}{\partial \dot{p}} = \frac{\dot{p}^2}{(C\,p+\dot{p})[C(1-p)-\dot{p}]} = K$$

K=1st integral of the motion

$$p(t) = \frac{1}{e^{\epsilon(t)/kT}+1}\left[1+\sqrt{K\,e^{\epsilon(t)/kT}}\right]$$

K measures the degree of dissipation.
The larger is K, the large is dissipation.

Carnot limit: $K \rightarrow 0$

Solution at Weak Dissipation Limit

$$K \ll 1$$

$$\Delta S_{\mathrm{H}} = \Delta S_{\mathrm{rev}} - \frac{[\phi_1 - \phi_0]^2}{C_{\mathrm{H}}} \frac{1}{\tau_{\mathrm{H}}} \qquad \Delta S_{\mathrm{C}} = -\Delta S_{\mathrm{rev}} - \frac{[\phi_1 - \phi_0]^2}{C_{\mathrm{C}}} \frac{1}{\tau_{\mathrm{C}}}$$

$$\phi_i = \arcsin(1 - 2 p_i) \qquad i = 0,1$$

$$P = \frac{Q_{\mathrm{H}} + Q_{\mathrm{C}}}{\tau_{\mathrm{H}} + \tau_{\mathrm{C}}} = \frac{(T_{\mathrm{H}} - T_{\mathrm{C}}) \Delta S_{\mathrm{rev}} - (\phi_1 - \phi_0)^2 \left[\frac{T_{\mathrm{H}}}{C_{\mathrm{H}} \tau_{\mathrm{H}}} + \frac{T_{\mathrm{C}}}{C_{\mathrm{C}} \tau_{\mathrm{C}}} \right]}{\tau_{\mathrm{H}} + \tau_{\mathrm{C}}}$$

$$\eta^* = 1 + \frac{Q_{\mathrm{C}}^*}{Q_{\mathrm{H}}^*} = \frac{\eta_{\mathrm{C}}}{2} + \frac{\eta_{\mathrm{C}}^2}{4\left(1 + \sqrt{\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}}}\right)} + \frac{\eta_{\mathrm{C}}^3}{8\left(1 + \sqrt{\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}}}\right)} + \cdots$$

If $C_{\mathrm{H}} = C_{\mathrm{C}} \rightarrow \eta^* = \frac{\eta_{\mathrm{C}}}{2} + \frac{\eta_{\mathrm{C}}^2}{8} + \frac{\eta_{\mathrm{C}}^3}{16} + \cdots = \eta_{\mathrm{CA}}$

Generalization

Change in system entropy

$$\Delta S = \Delta S^{\mathrm{e}} + \Delta S^{\mathrm{i}}$$

Internal entropy production

Entropy flow: $\Delta S^{\mathrm{e}} = \dfrac{Q}{T}$

Quasi static limit $\quad \tau \to \infty: \ \Delta S^{\mathrm{e}} \to \Delta S^{\mathrm{rev}}, \quad \Delta S^{\mathrm{i}} \to 0$

Asymptotic expansion (Weak dissipation approximation)

$$\Delta S^{\mathrm{e}} = \Delta S^{\mathrm{rev}} - \frac{\Sigma^{\mathrm{e}}}{\tau} + o\left(\frac{1}{\tau^2}\right)$$

$$\Delta S^{\mathrm{i}} = \frac{\Sigma^{\mathrm{i}}}{\tau} + o\left(\frac{1}{\tau^2}\right)$$

Heat

$$Q_H = T_H \Delta S_H^e \approx T_H \Delta S^{rev} - \frac{T_H \Sigma_H^e}{\tau_H}$$

$$Q_C = T_C \Delta S_C^e \approx -T_C \Delta S^{rev} - \frac{T_C \Sigma_C^e}{\tau_C}$$

Power

$$P = \frac{Q_H + Q_C}{\tau_H + \tau_C} = \frac{(T_H - T_C)\Delta S^{rev} - \frac{T_H \Sigma_H^e}{\tau_H} - \frac{T_C \Sigma_C^e}{\tau_C}}{\tau_H + \tau_C}$$

Maximizing power with respect to τ_H and τ_C

$$\tau_H^* = \frac{2 T_H \Sigma_H^e}{(T_H - T_C)\Delta S^{rev}} \left(1 + \sqrt{\frac{T_C \Sigma_C^e}{T_H \Sigma_H^e}}\right)$$

$$\tau_C^* = \frac{2 T_C \Sigma_C^e}{(T_H - T_C)\Delta S^{rev}} \left(1 + \sqrt{\frac{T_H \Sigma_H^e}{T_C \Sigma_C^e}}\right)$$

Efficiency at *Maximum* Power

$$\eta^* = 1 + \frac{Q_C^*}{Q_H^*} = \frac{\eta_C}{2} + \frac{\eta_C^2}{4\left(1+\sqrt{\frac{\Sigma_C^e}{\Sigma_H^e}}\right)} + \frac{\eta_C^3}{8\left(1+\sqrt{\frac{\Sigma_C^e}{\Sigma_H^e}}\right)} + \cdots$$

$$\frac{\Sigma_C^e}{\Sigma_H^e} = \infty \rightarrow \eta^* = \frac{\eta_C}{2}$$

$$\frac{\Sigma_C^e}{\Sigma_H^e} = 0 \rightarrow \eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{4} + \frac{\eta_C^3}{8} + \cdots = \frac{\eta_C}{2 - \eta_C}$$

$$\frac{\Sigma_C^e}{\Sigma_H^e} = 1 \rightarrow \eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{16} + \cdots = \eta_{CA}$$

$$\frac{\eta_C}{2} \leq \eta^* \leq \frac{\eta_C}{2 - \eta_C}$$

$\uparrow$

η_{CA}

Exactly in the middle of two bounds

I think I have already said so.

Schmiedl & Seifert (2008)

TABLE I. Observed performance of real heat engines.

Power source	T_C	T_H	η_C	η_{CA}	$\eta_{Observed}$	η_L (Lower bound)	η_U (Upper bound)
West Thurrock (U.K.)[2] Coal Fired Steam Plant	~25	565	64.1%	40%	36%	32%	47%
CANDU (Canada)[4] PHW Nuclear Reactor	~25	300	48.0	28%	30%	24%	31%
Larderello (Italy)[5] Geothermal Steam Plant	80	250	32.3%	17.5%	16%	16%	19%
Steam power plant (USA)	298	923	67.6%	43.2%	40%	34%	51%

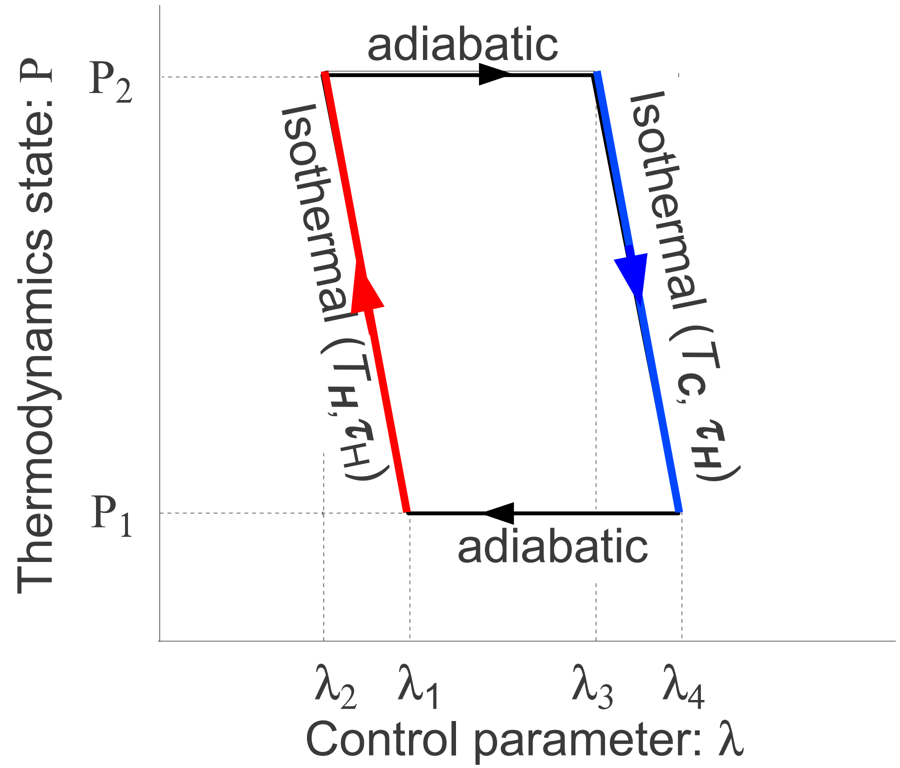
Justification of asymptotic expansion

An isothermal process: a master equation

$$\dot{P}_m(t)=\sum_n W^T_{mn}(t)P_n(t)$$

$$W^T_{mn}(t)=W^T_{mn}[\lambda(t/\tau)]$$

$$\sum_m W^T_{mn}(t)=0$$



Scaling time

$$\frac{d}{ds}|P_\tau(s)\rangle=\tau\,\hat{W}^T(s)|P_\tau(s)\rangle \qquad s=\frac{t}{\tau}$$

Asymptotic expansion

$$|P_\tau(s)\rangle=|P^{st}(s)\rangle+\frac{1}{\tau}|\delta^{(1)}(s)\rangle+\frac{1}{\tau^2}|\delta^{(2)}(s)\rangle+o\left(\frac{1}{\tau^3}\right)$$

$$\hat{W}^T(s)|P^{st}(s)\rangle=0,\ \text{Detailed balance}\quad W^T_{mn}(s)P^{st}_n(s)=W^T_{nm}(s)P^{st}_m(s)$$

$$\hat{W}^T(s)|\delta^{(1)}(s)\rangle=\frac{d}{ds}|P^{st}(s)\rangle,\quad \hat{W}^T(s)|\delta^{(2)}(s)\rangle=\frac{d}{ds}|\delta^{(1)}(s)\rangle \qquad \sum_m\delta^{(1)}_m=\sum_m\delta^{(2)}_m=0$$

Entropy flow

$$\Delta S^{\mathrm{e}} = \int_0^{\tau} dt \sum_m \sum_n W_{mn}(t) P_n(t) \ln\left[\frac{W_{nm}(t)}{W_{mn}(t)}\right]$$

$$= \Delta S^{\mathrm{rev}} - \frac{1}{\tau}\int_0^1 ds \sum_m \left[\frac{d}{ds}\delta_m^{(1)}(s)\right] \ln P_m^{\mathrm{st}}(s)$$

$$\Delta S^{\mathrm{rev}} = -k \sum_m P_m^{\mathrm{st}}(\tau) \ln P_m^{\mathrm{st}}(\tau) + k \sum_m P_m^{\mathrm{st}}(0) \ln P_m^{\mathrm{st}}(0)$$

Entropy production

$$\Delta S^{\mathrm{i}} = \int_0^{\tau} dt \sum_m \sum_n W_{mn}(t) P_n(t) \ln\left[\frac{W_{mn}(t) P_n(t)}{W_{nm}(t) P_m(t)}\right]$$

$$= -\frac{1}{\tau}\int_0^1 ds \sum_m \delta_m^{(1)}(s) \left[\frac{d}{ds} \ln P_m^{\mathrm{st}}(s)\right]$$

Conclusions

1) The efficiency at maximum power is derived without a specific model at the weak dissipation limit, .
2) The minimum efficiency at maximum power is $\eta_C/2$. (If only dissipation due to finite time is taken into account.)
3) The maximum efficiency at maximum power is $\eta_C/(2-\eta_C)$
4) Exact Curzon-Ahlborn efficiency is obtained when left-right symmetry holds and it lies right in the middle between the lower and upper bounds.
5) Only maximization of power with respect to operation times is necessary to get the Curzon-Ahlborn efficiency
6) The method of asymptotic expansion (weak dissipation limit) is justified for general Markovian processes.
7) The general results are demonstrated using analytically solvable model based on a quantum dot Carnot engine.

Determination of K

$$C_\alpha \tau_\alpha = F(p(\tau_\alpha), K) - F(p(0), K)$$

$$F(p, K) = \frac{-1}{2} \ln p + \frac{1}{\sqrt{K}} \arctan\left[\frac{1-2p}{\sqrt{K+4p(1-p)}}\right] + \frac{1}{2} \ln\left[\frac{2p + K + \sqrt{K^2 + 4p(1-p)}}{2(1-p) + K + \sqrt{K^2 + 4Kp(1-p)}}\right]$$

Exact Entropy flow

$$Q = T\,\Delta S^e = \mathcal{S}(p(\tau), K) - \mathcal{S}(p(0), K)$$

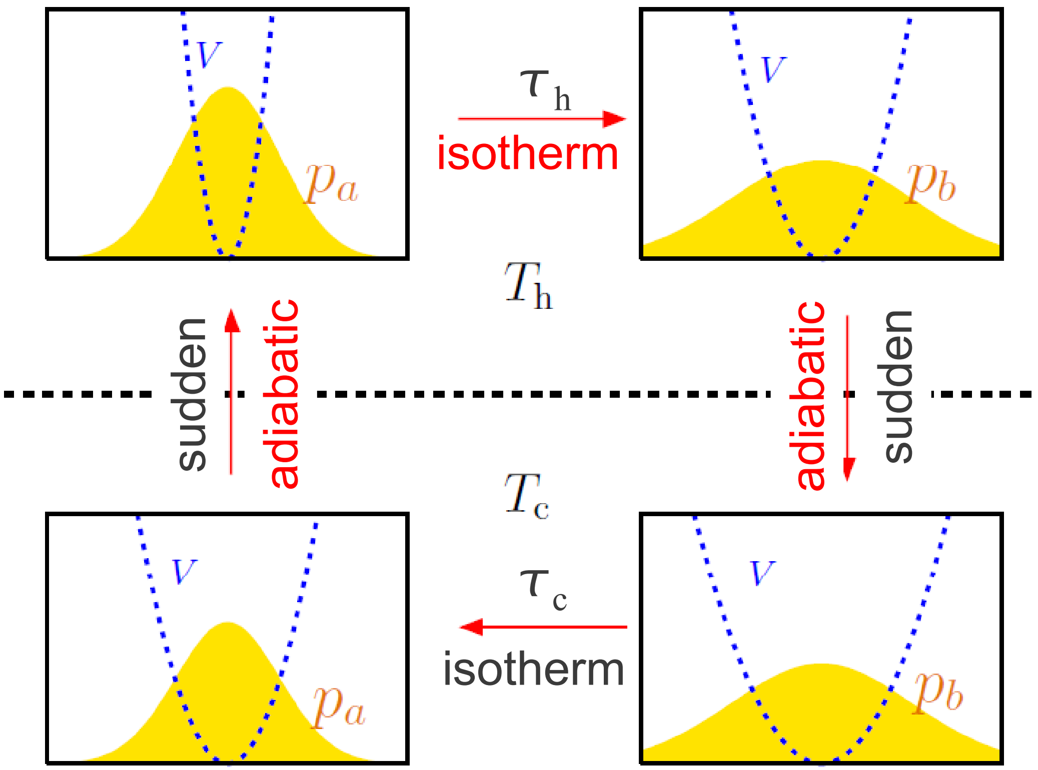
$$\mathcal{S}(p, K) = p \ln\left[\frac{2(1-p)\,p + K - \sqrt{K^2 + 4Kp(1-p)}}{2p^2}\right] - \sqrt{K} \arcsin\left[\frac{1-2p}{\sqrt{K+1}}\right] - \ln\left[\frac{2(1-p) - K - \sqrt{K^2 + 4Kp(1-p)}}{2}\right]$$

Protcols

$$p_{\mathrm{C}}(t) = \frac{1}{2}\left[1 - \sin\left(\frac{t}{\tau_{\mathrm{C}}}\left|\phi_1 - \phi_0\right| + \phi_0\right)\right]$$

$$p_{\mathrm{H}}(t) = \frac{1}{2}\left[1 + \sin\left(\frac{t}{\tau_{\mathrm{H}}}\left|\phi_1 - \phi_0\right| - \phi_1\right)\right]$$

Brownian heat engine



Controle parameter: spring constant

Thermodynamic state: density $p(x)$

$$\eta^* = \frac{2(T_h - T_c)}{3T_h + T_c} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{32} + \ldots$$

Schmiedl & Seifert (2008)