Momentum Deficit due to Dissipation
- From Hydrodynamics Perspective -

(What is the non-equilibrium force on a Brownian particle and how does the environment generate it?)

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arXiv:1301.7035

Stochastic Thermodynamics (NORDITA, 3/4-3/15, 2013)

Contents

- Introduction: Langevin force and Stochastic Thermodynamics
- Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.
  - Examples:
    - Brownian ratchet, Adiabatic piston,
    - Inelastic Piston, Granular ratchet
- Momentum Deficit due to Dissipation (MDD) is introduced.
- Force caused by MDD explains it all.
- The environment adjusts itself to MDD and it is no longer equilibrium.
- Sasa's Paradox

What is the force on a Brownian object exerted by an *environment*?

If the environment is an ideal heat bath (energy reservoir at equilibrium)

$$M\dot{V} = -\gamma V + \sqrt{2\gamma k_B T}\,\xi - \nabla U(X)$$

$$\langle \xi(t) \rangle = 0$$

$$M\langle \dot{V} \rangle = -\gamma \langle V \rangle - \langle \nabla U(X) \rangle$$

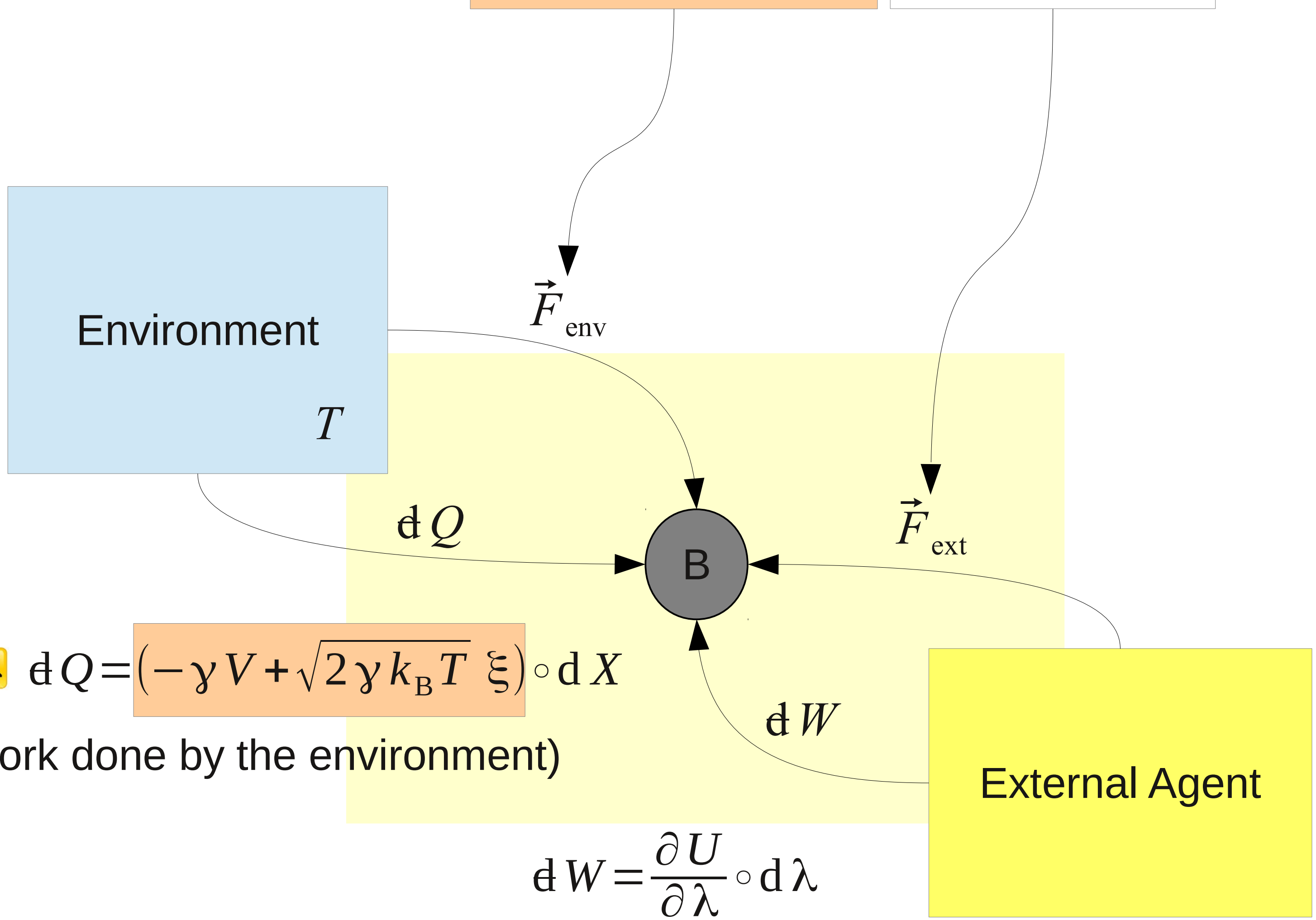
$$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$$

What is the force on a Brownian object exerted by the environment under a non-equilibrium condition?

Is the usual heat bath good enough?
If not, is there a universal environment with *desired properties*?

Stochastic Thermodynamics: Driven Non-equilibrium Processes

$$M\dot{V} \;=\; -\nabla U(X) \;\; -\gamma V + \sqrt{2\gamma k_B T}\;\xi \;\; -\nabla U_{\text{ext}}(X,\lambda)$$



🙂 $\text{đ}Q = \left(-\gamma V + \sqrt{2\gamma k_B T}\;\xi\right)\circ \mathrm{d}X$

(work done by the environment)

$$\text{đ}W = \frac{\partial U}{\partial \lambda}\circ \mathrm{d}\lambda$$

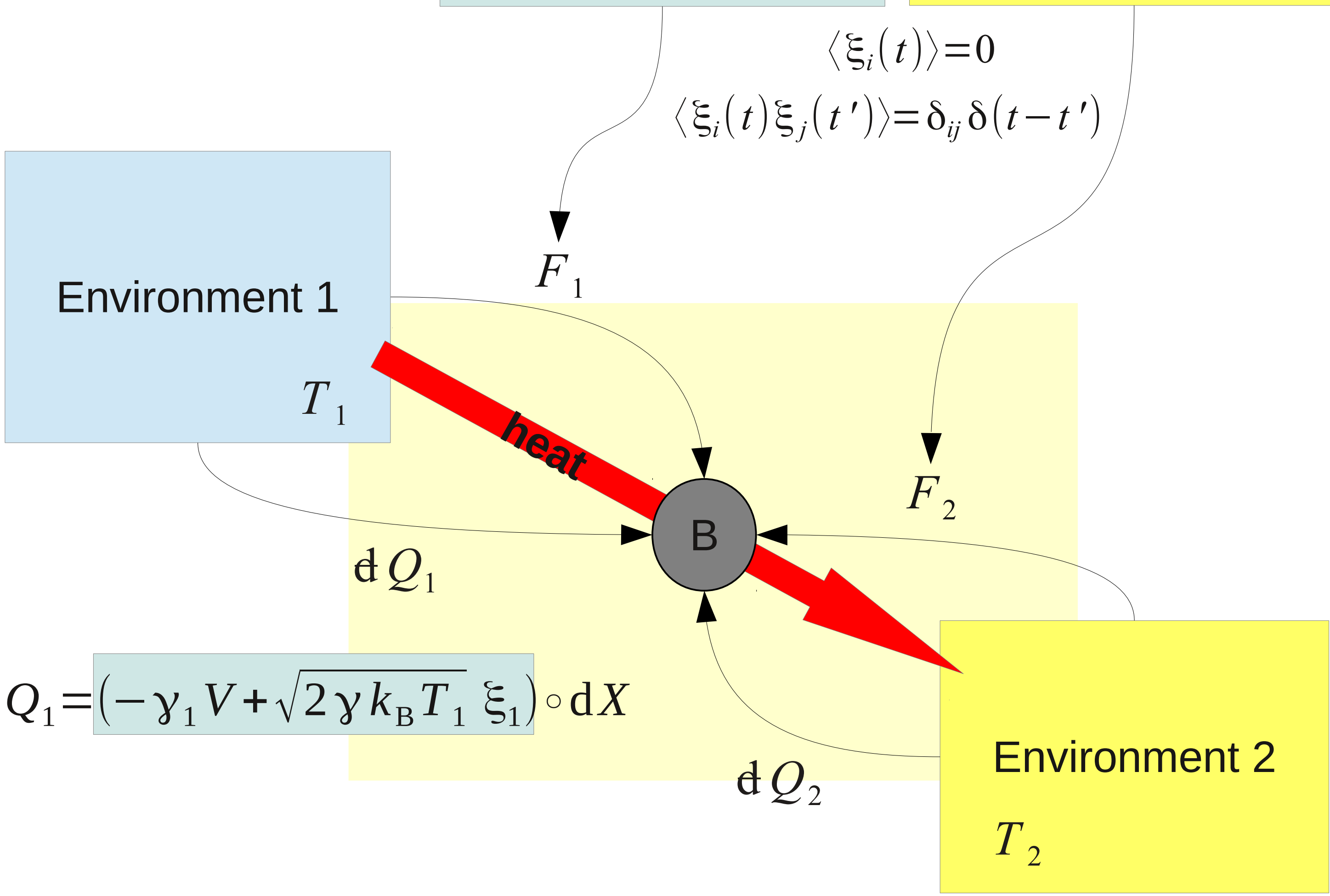
Sekimoto, *Stochastic Energetics* (Springer, 2010).

Stochastic Thermodynamics: Non-equilibrium Steady State (NESS)

$$M\dot{V} = -\nabla U(X) \quad -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 \quad -\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2$$

$$\langle \xi_i(t) \rangle = 0$$

$$\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\,\delta(t-t')$$

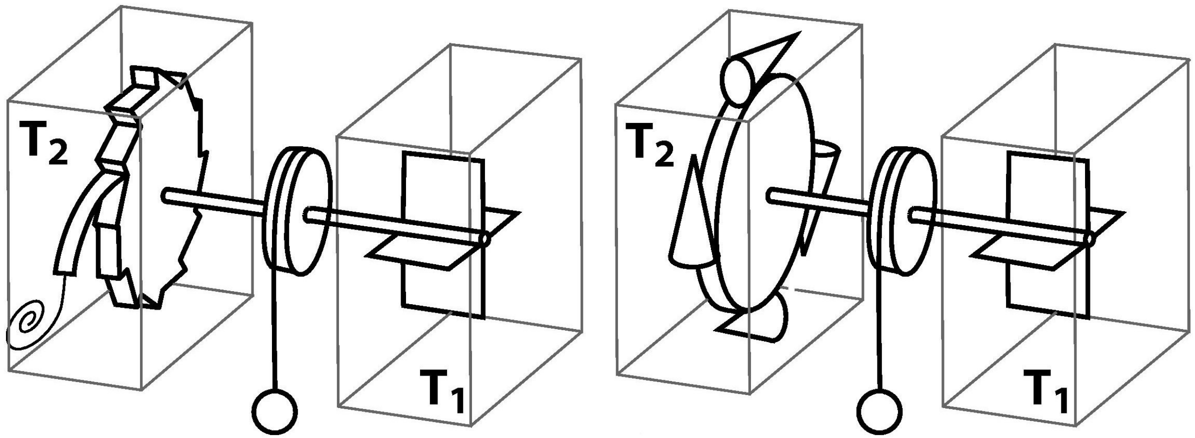


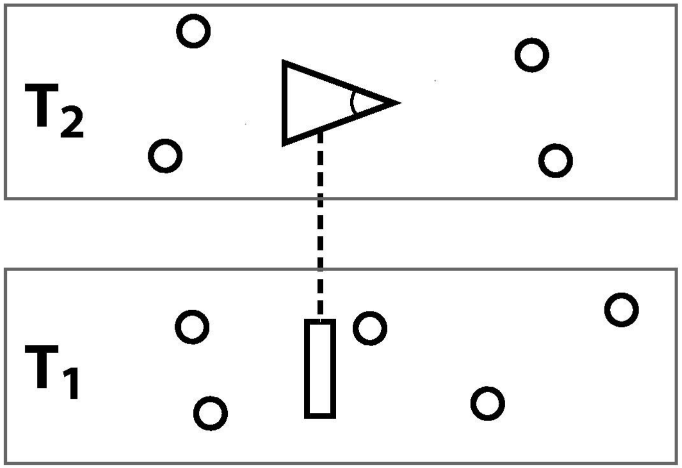
$$đQ_1 = \left(-\gamma_1 V + \sqrt{2\gamma k_B T_1}\,\xi_1\right) \circ dX$$

$$đQ_2 = \left(-\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2\right) \circ dX$$

Sekimoto, *Stochastic Energetics* (Springer, 2010).

Smoluchowski-Feynman Ratchet

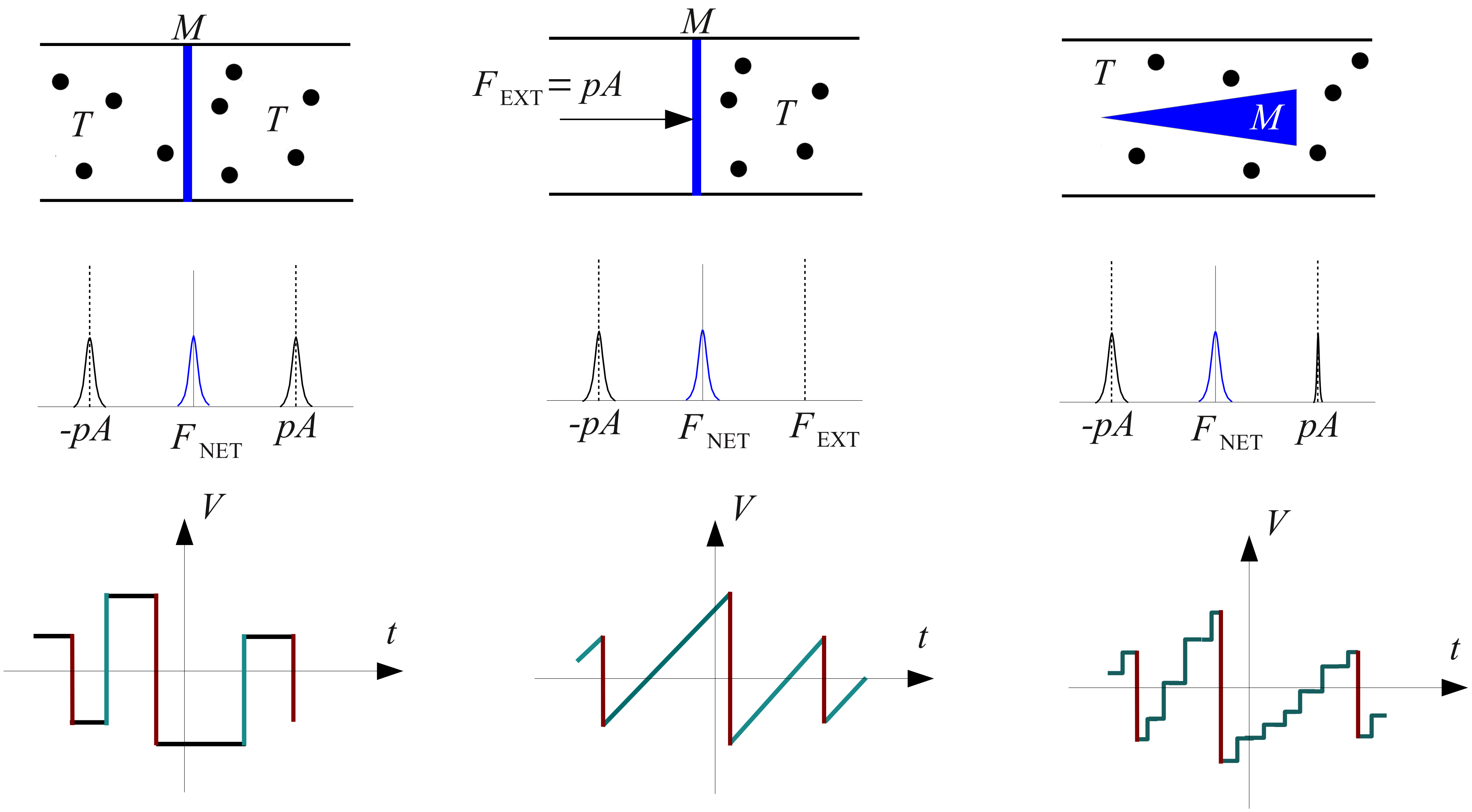




Van den Broeck, R.K. and Meurs(2004), PRL **93** 090601

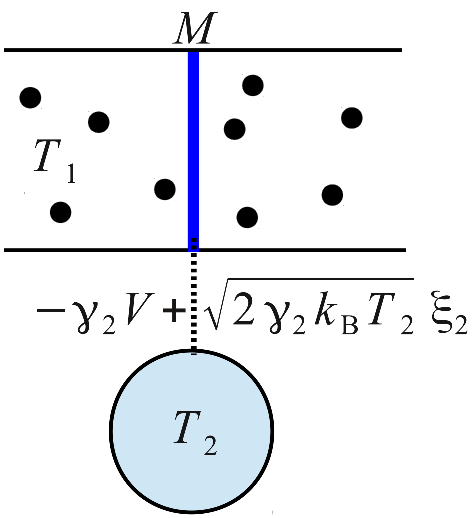
Lagevin Eq. can't see asymmetric situation around the Brownian object.
(Coarse graining hides it.)

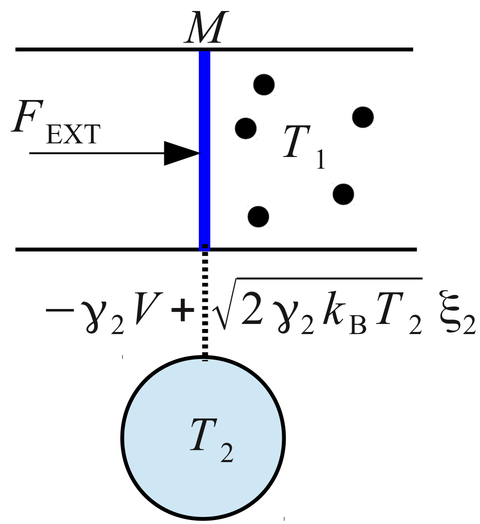
$$M\dot{V} = -\gamma V + \sqrt{2\gamma k_B T}\,\xi + F_{EXT} \qquad \langle V \rangle = 0, \qquad \langle V^2 \rangle = \frac{k_B T}{M}$$

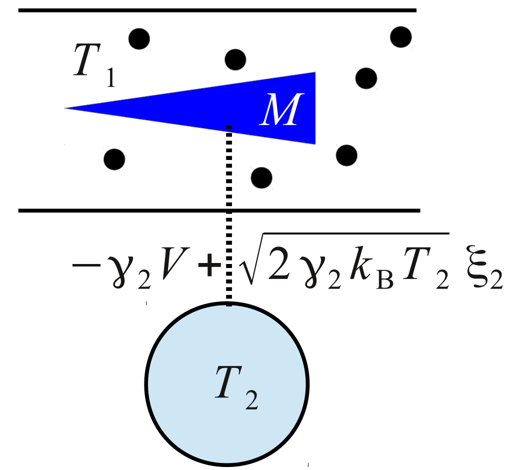


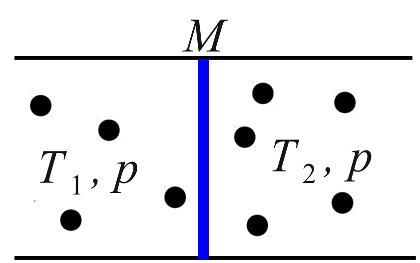
Adding a second "heat bath" illuminates the asymmetry.
How?

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 + F_{EXT}$$



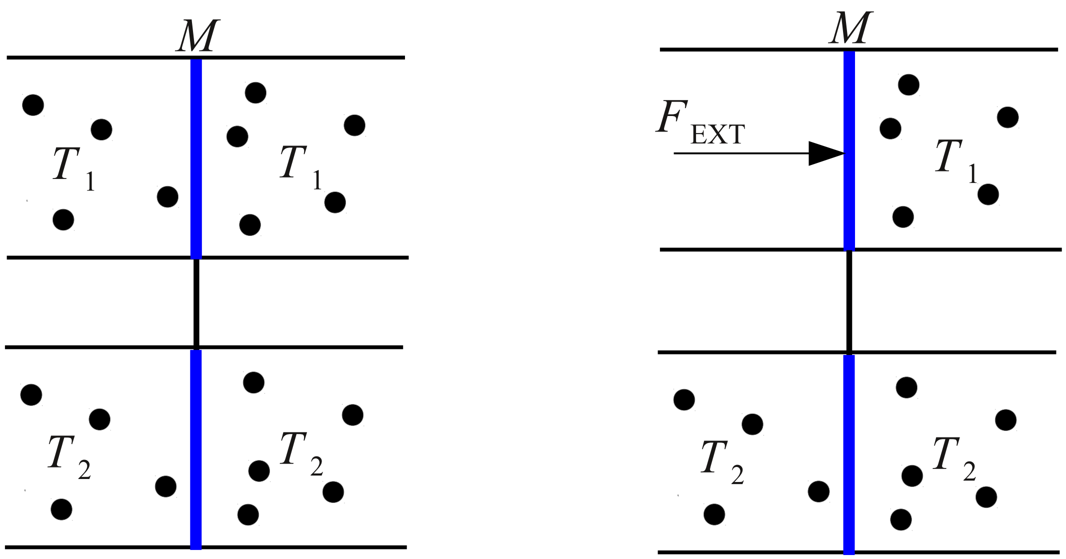




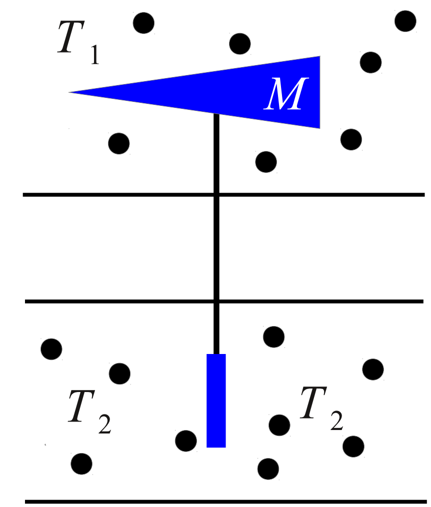


Replace a “ideal” heat bath with a “realistic reservoir”.

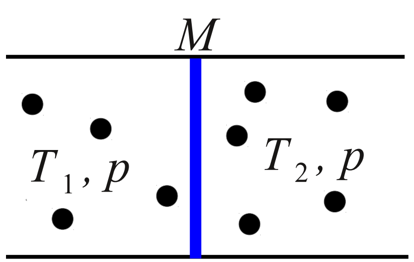
Shared pistons



Ratchet

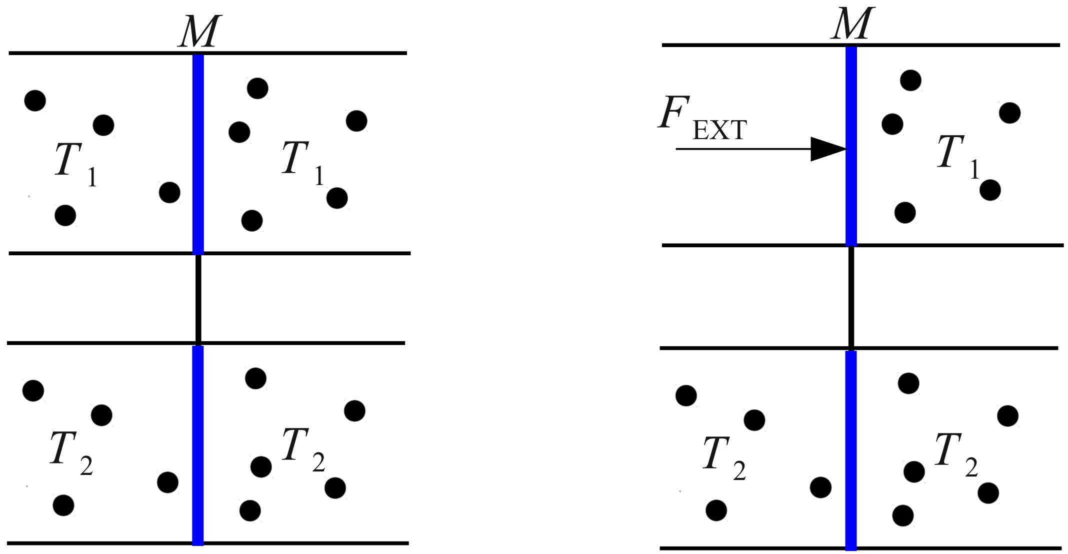


Adiabatic piston

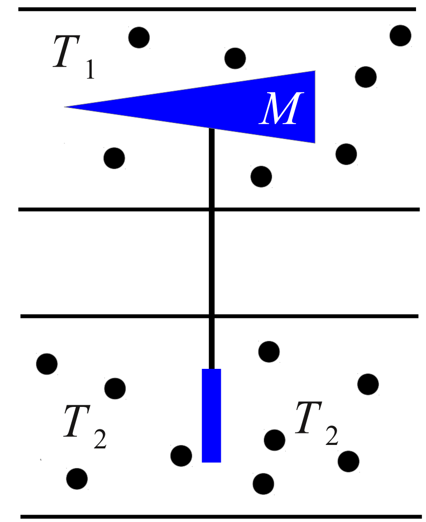


Replace a “ideal” heat bath with a “realistic reservoir”.

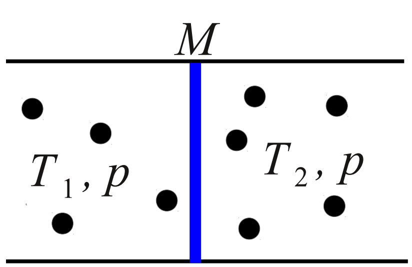
Shared pistons



Ratchet

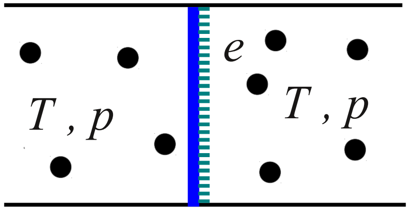


Adiabatic piston



Granular Brownian objects

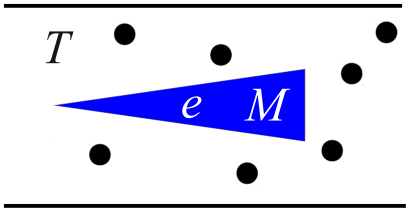
Inelastic piston



$e<1$

Costantini et al. EPL 82,50008 (2008)

Granular ratchet



$e<1$

Costantini et al. PRE 75, 061124 (2007)
Cleuren and Van den Broeck, EPL 77, 50003 (2007)
Talbot et al. PRE 82, 011135 (2010)

Internal degrees act as the second environment.

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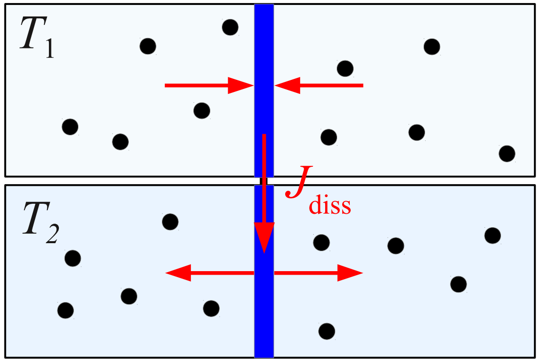
Shared Brownian piston



$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 \quad -\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2$$

$$M\langle\dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle, \qquad \langle V\rangle \rightarrow 0$$

Heat flow through the fluctuation of the pistons' velocity.

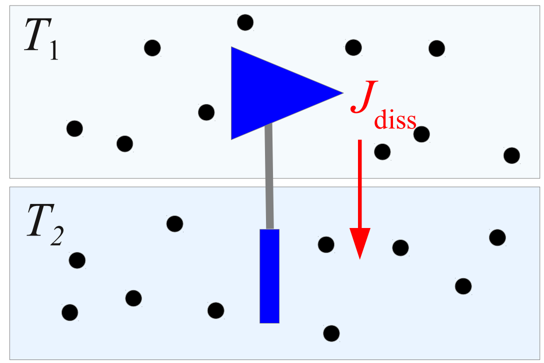
$$\frac{M}{2}\langle V^2\rangle = \frac{k_B T_{\rm kin}}{2}, \quad \text{where} \quad T_{\rm kin} = \frac{\gamma_1 T_1 + \gamma_2 T_2}{\gamma_1+\gamma_2}$$

$$J_{\rm diss} = \frac{kT_1 - kT_2}{M\left(\gamma_1^{-1}+\gamma_2^{-1}\right)}$$

☺ Stochastic Thermodynamics

Sekimoto (1997), Parrondo and Española (1996), Van den Breock *et. al.* (2001)

Brownian Motor



It moves!

$$\langle V\rangle > 0 \quad (T_1 < T_2)$$

$$\langle V\rangle < 0 \quad (T_1 > T_2)$$

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 \quad -\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2$$

$$M\langle\dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle, \qquad \langle V\rangle \to 0$$

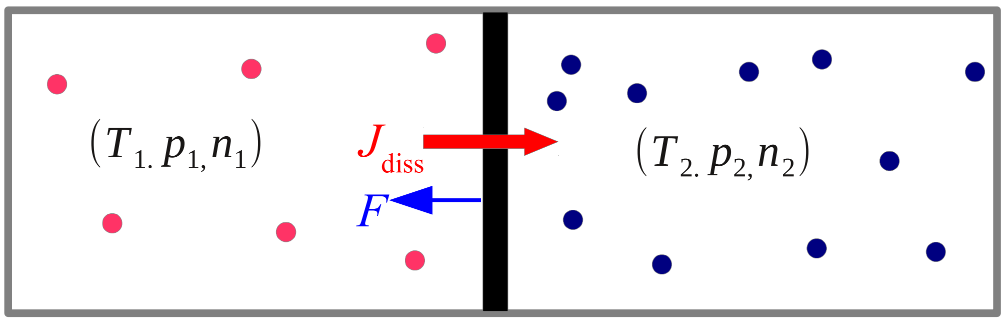
$$M\langle\dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle + c\left(\langle V^2\rangle - \frac{k_B T_1}{M}\right)$$

Boltzmann-master eq. predicts this additional force.

$$\frac{M}{2}\langle V^2\rangle = \frac{k_B T_{\text{kin}}}{2} \quad \rightarrow \quad c\left(\frac{k_B T_{\text{kin}}}{M} - \frac{k_B T_1}{M}\right) \propto J_{\text{diss}}$$

Van den Broeck, R.K. and Meurs(2004), PRL **93** 090601

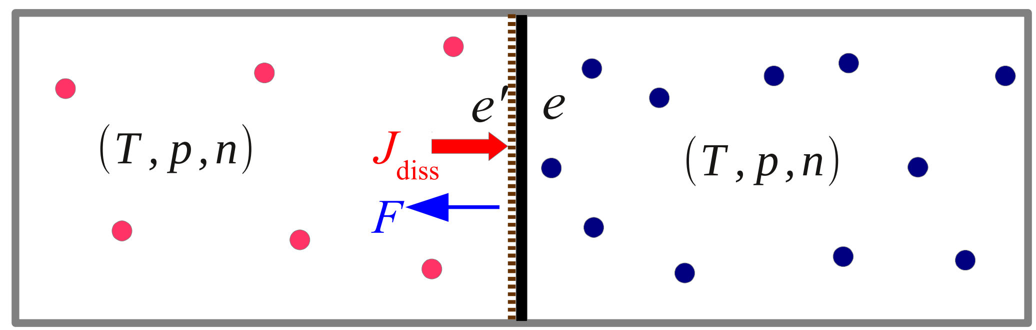
Adiabatic Piston



$p_1 = p_2\,,\quad T_1 > T_2\,,\quad n_1 < n_2$

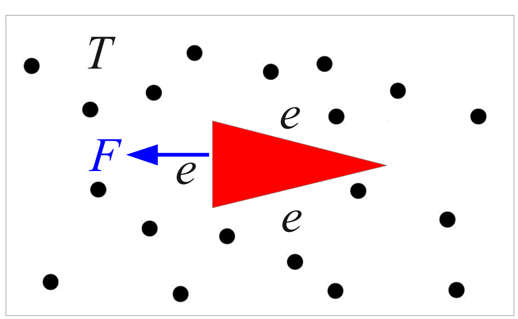
$p_i = n_i k_B T_i$

Inelastic Piston



$e' < e$

Granular ratchet



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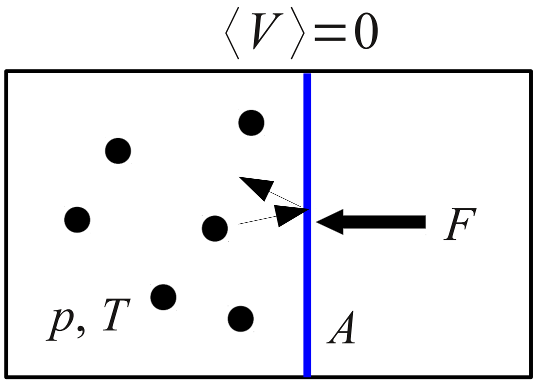
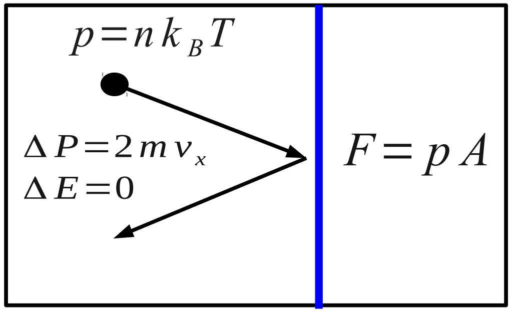
Is force on a wall $F = pA$?

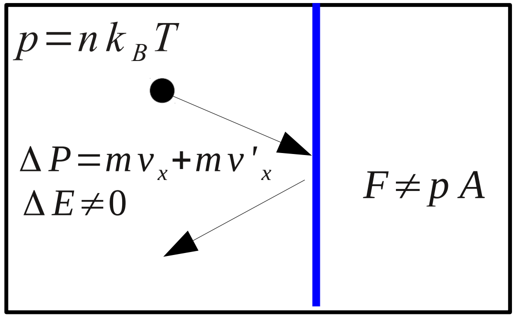


p = hydrostatic pressure

Thermally Equilibrium

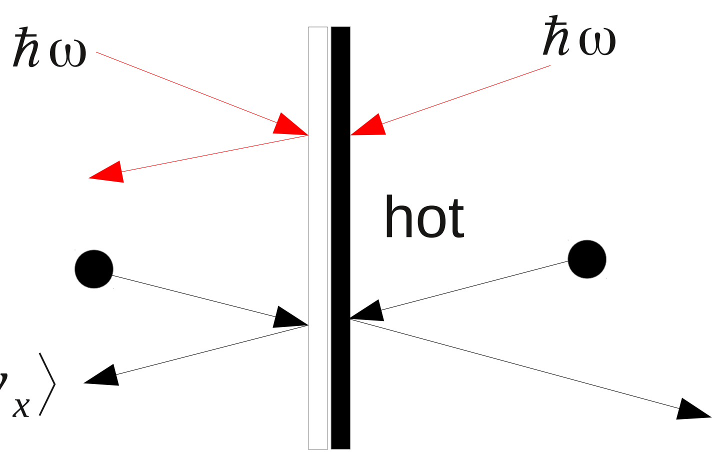


Thermally Non-equilibrium



Macroscopic example: Radiometer

J. Clerk Maxwell
Phil. Trans. R. Soc. Lond. **170** (1879), 231



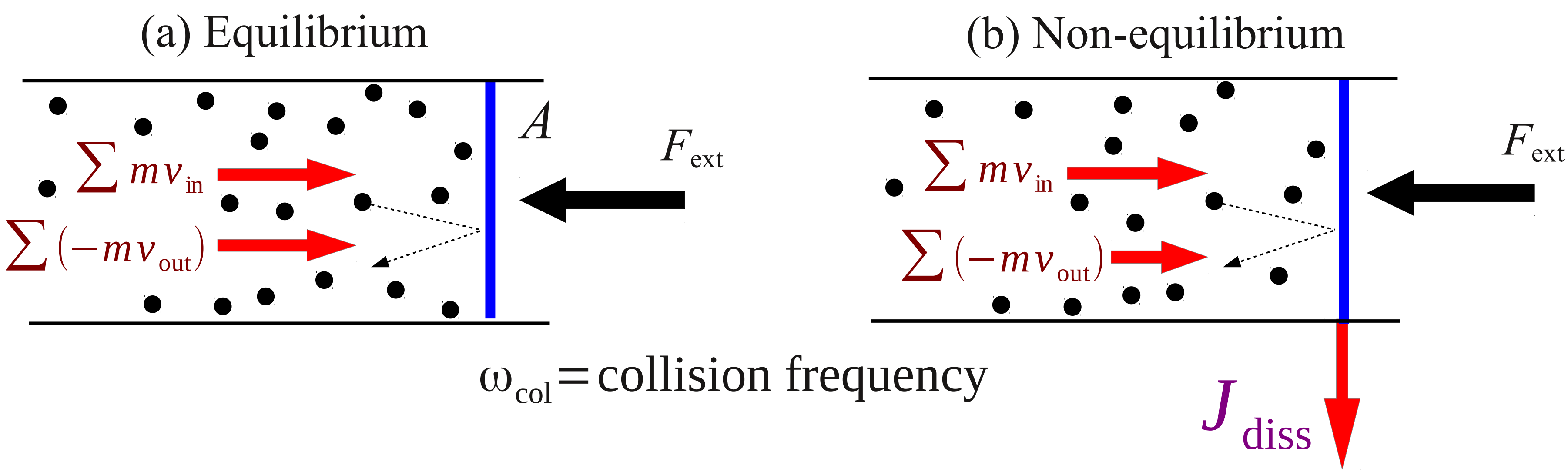
$$\langle \Delta E \rangle = 0$$

$$\langle \Delta P \rangle = 2m\langle v_x \rangle$$

$$\langle \Delta E \rangle > 0$$

$$\langle \Delta P \rangle = m\langle v_x \rangle + m\langle v'_x \rangle > 2m\langle v_x \rangle$$

Momentum Deficit due to Dissipation (MDD)



$$-F = (m v_{th} + m|v'|)\omega_{col} = (2 m v_{th} + m|v'| - m v_{th})\omega_{col} = pA + F_{MDD}$$

$$F_{MDD} = (m|v'| - m v_{th})\omega_{col} \qquad pA = 2 m v_{th}\omega_{col}$$

$$\left(\frac{1}{2} m v_{th}^2 - \frac{1}{2} m|v'|^2\right)\omega_{col} = J_{diss} \xrightarrow[v_{th} \sim |v|]{} (m v_{th} - m|v'|)\omega_{col} \approx \frac{J_{diss}}{v_{th}}$$

This agrees with the result of lengthy calculation of Boltzmann-Master eq. →

$$F_{MDD} \approx -c\frac{J_{diss}}{v_{th}}$$

$c = \sqrt{\frac{\pi}{8}}$ for hard disk gas

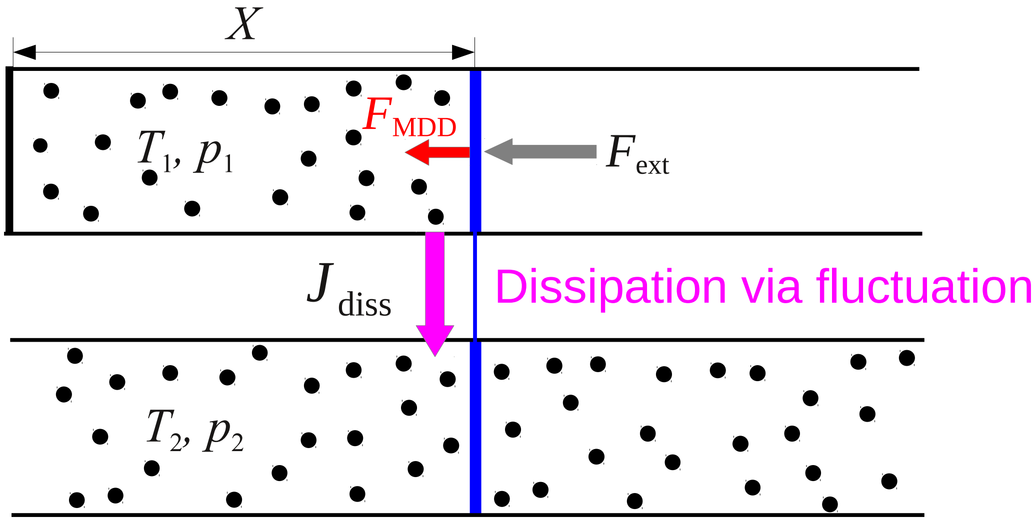
Claim

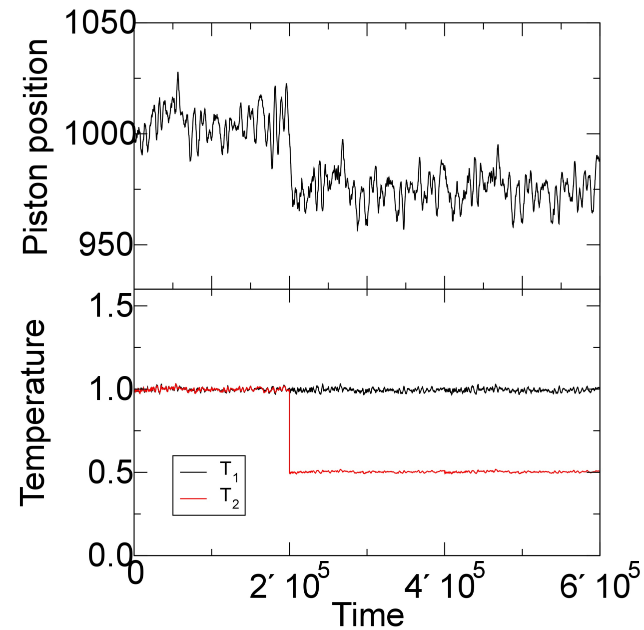
- When heat dissipates through the motion of a Brownian object, the environment exerts a force on the Brownian object.
- Its direction is opposite to the direction of the heat flux.
- Its magnitude is proportional to the heat flux, more specifically

$$F_{\mathrm{MDD}} \approx -c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}} \qquad c > 0$$

$$M \langle \dot{V} \rangle = -(\gamma_1 + \gamma_2) \langle V \rangle - c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

Simple Model 1: Shared Brownian Piston

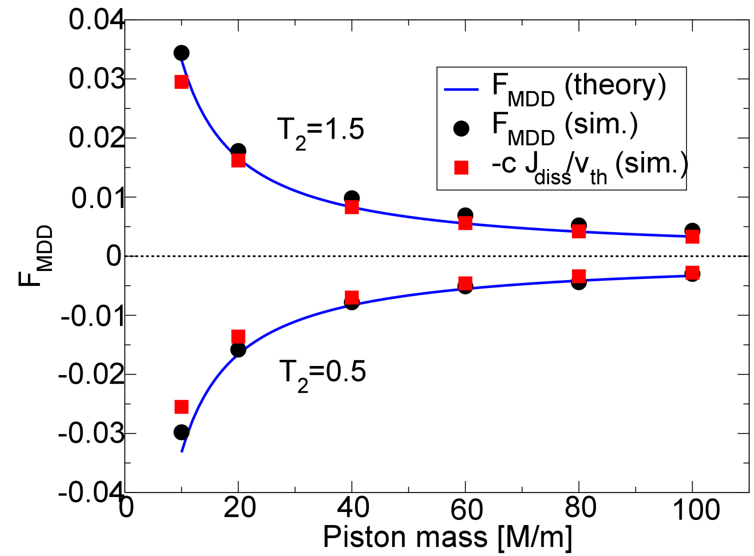




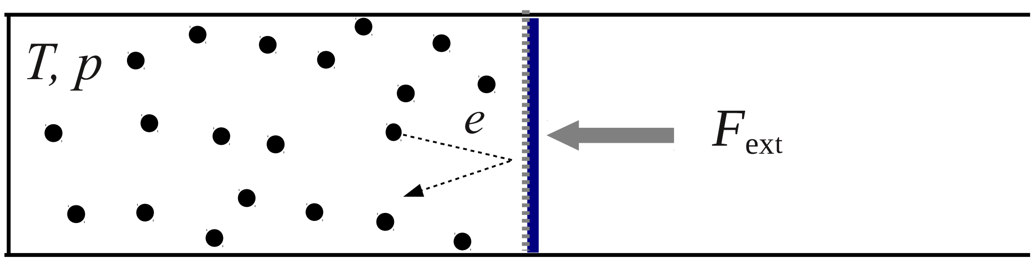
$$F_{\mathrm{MDD}} = -\sqrt{\frac{\pi}{8}}\frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

$$J_{\mathrm{diss}} = \sqrt{\frac{\pi}{8}}\frac{kT_1 - kT_2}{M\left(\gamma_1^{-1} + \gamma_2^{-1}\right)}$$

$$\frac{F_{\mathrm{MDD}}}{L} = -\frac{2\rho_1\rho_2}{\rho_1 + 2\rho_2}\frac{m}{M}\left(kT_1 - kT_2\right)$$



Simple Model 2: Inelastic Brownian Piston



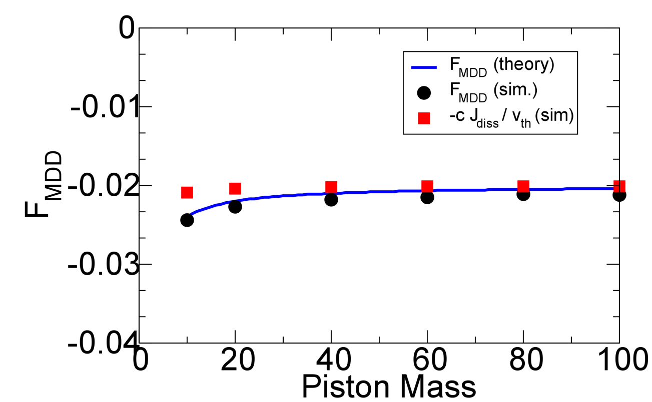
e=restitution coefficient

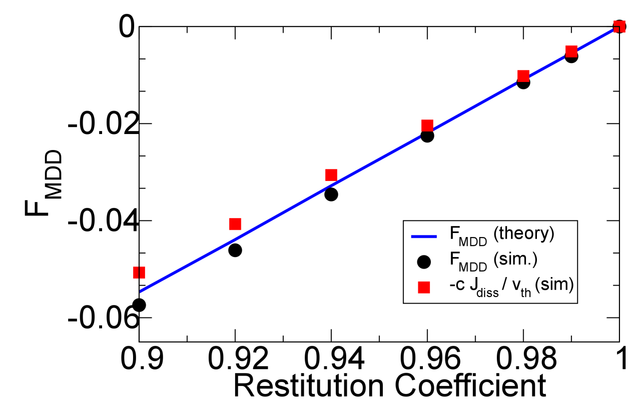
House keeping dissipation

$$J_{\text{diss, hk}} = (1-e)\sqrt{\frac{2}{\pi}}\, v_{\text{th}}\, p\, L \longrightarrow \frac{F_{\text{MDD,hk}}}{L} = -\frac{1}{2}(1-e)\, p$$

Excess dissipation

$$J_{\text{diss, ex}} = (1-e)\frac{\gamma}{M}\, v_{\text{th}}\, p\, L \longrightarrow \frac{F_{\text{MDD,hk}}}{L} = -\frac{m}{M}(1-e)\, p$$





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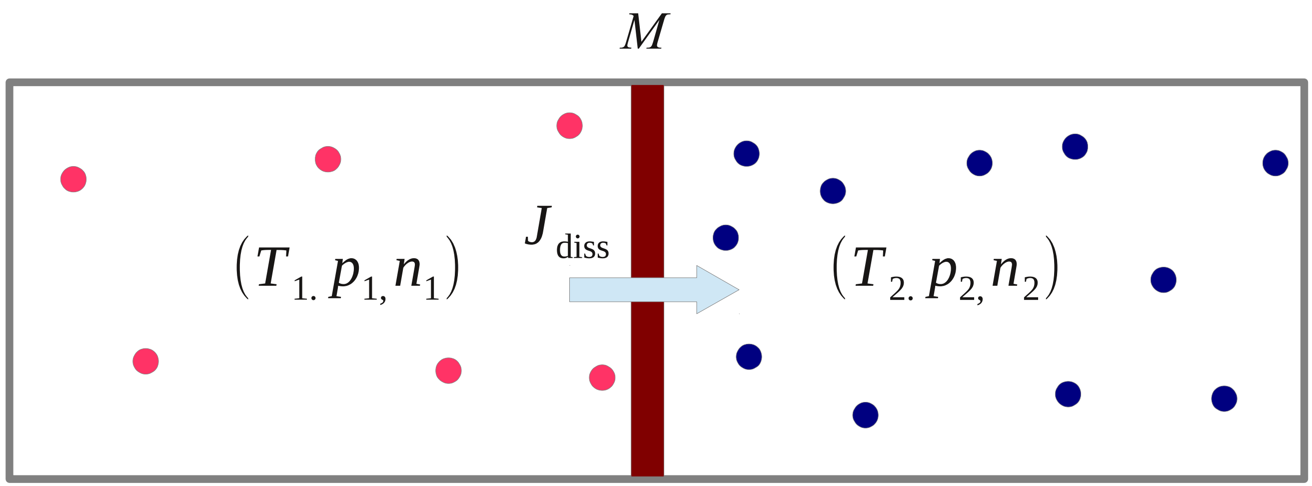
Adiabatic Piston



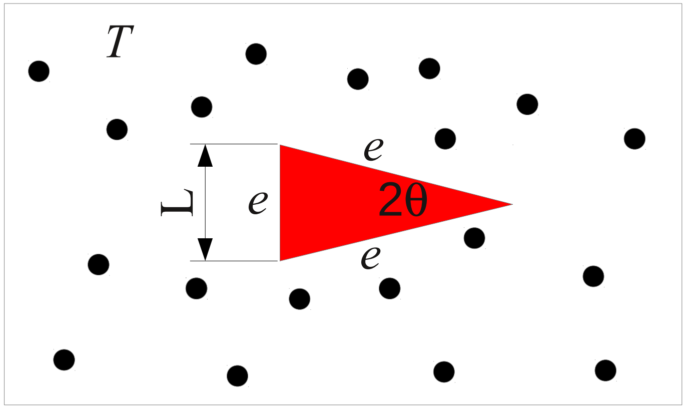
$p_1 = p_2 \,, \quad T_1 > T_2 \,, \quad n_1 < n_2$

$$F_{\mathrm{MDD},1} = -\sqrt{\frac{\pi}{8}}\,\frac{J_{\mathrm{diss}}}{v_{\mathrm{th},1}} \qquad F_{\mathrm{MDD},2} = \sqrt{\frac{\pi}{8}}\,\frac{(-J_{\mathrm{diss}})}{v_{\mathrm{th},2}}$$

$$F_{\mathrm{NET}} = -\sqrt{\frac{\pi}{8}}\,J_{\mathrm{diss}}\left(\frac{1}{v_{\mathrm{th},2}} + \frac{1}{v_{\mathrm{th},1}}\right) = \sqrt{\frac{\pi}{8}}\,\frac{kT_1 - kT_2}{M\left(\gamma_1^{-1} + \gamma_2^{-1}\right)}\left(\frac{1}{v_{\mathrm{th},1}} + \frac{1}{v_{\mathrm{th},2}}\right)$$

The piston moves in the opposite direction of the heat.

Granular Brownian Ratchet



$$F_{\text{BASE}} = P\,L - \frac{1-e}{2} P\,L - \frac{m}{M}(1-e)\,p\,L$$

$$F_{\text{SIDE}} \approx -P\,L + \frac{1-e}{2} P\,L + o(\theta)$$ (Fluctuation is negligible)

$$F_{\text{NET}} = -\frac{m}{M}(1-e)\,p\,L < 0$$

Cleuren and Van den Broeck, EPL **77** (2007) 50003

Costantini et al., Phys. Rev. E **75** (2007), 061124

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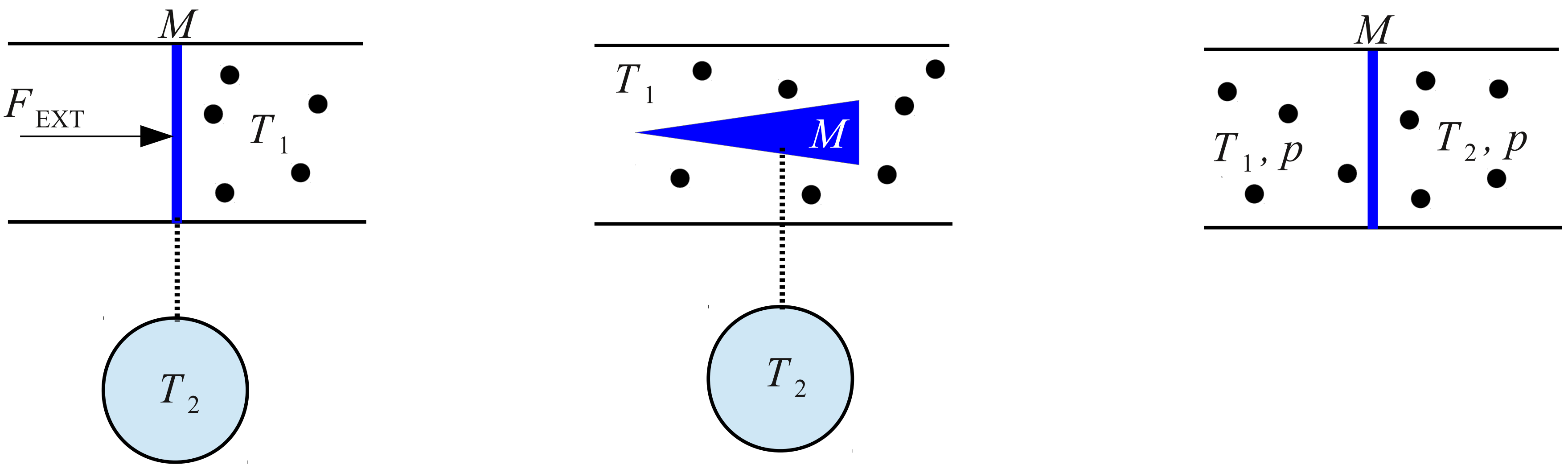
What kind of reservoir provides F_{MDD} ?

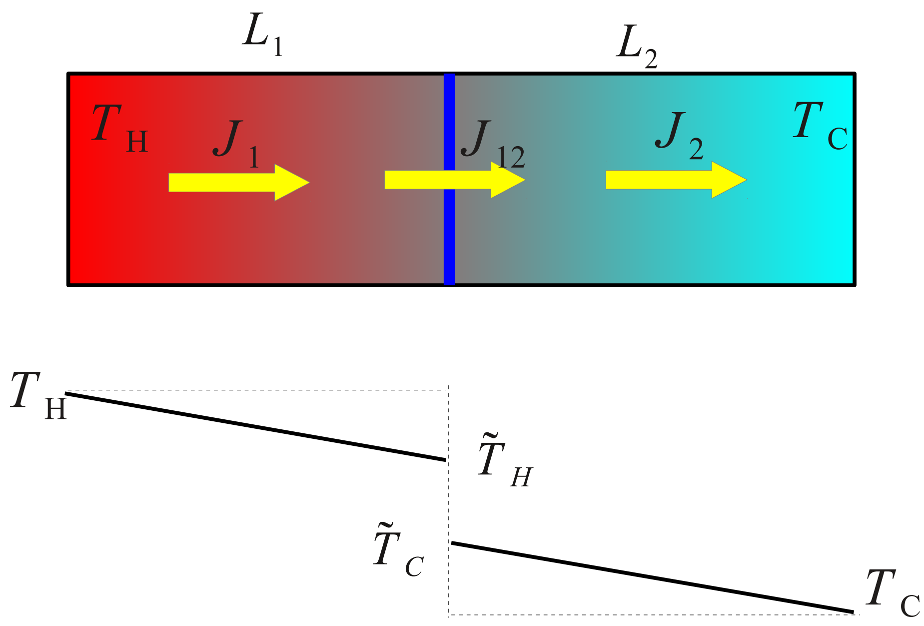


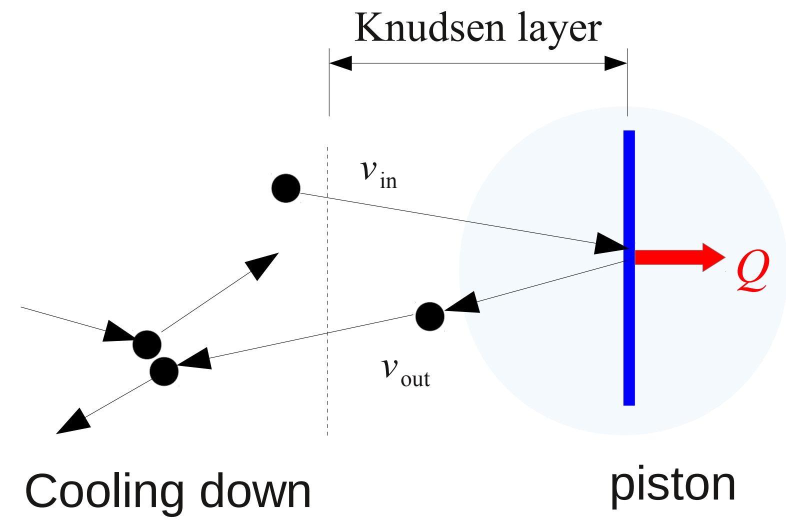
$$M\langle \dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle - c\frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 - \gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2 +$$ Non-linear ?

Plyukhin and Schofield, PRE 69, 021112 (2004)

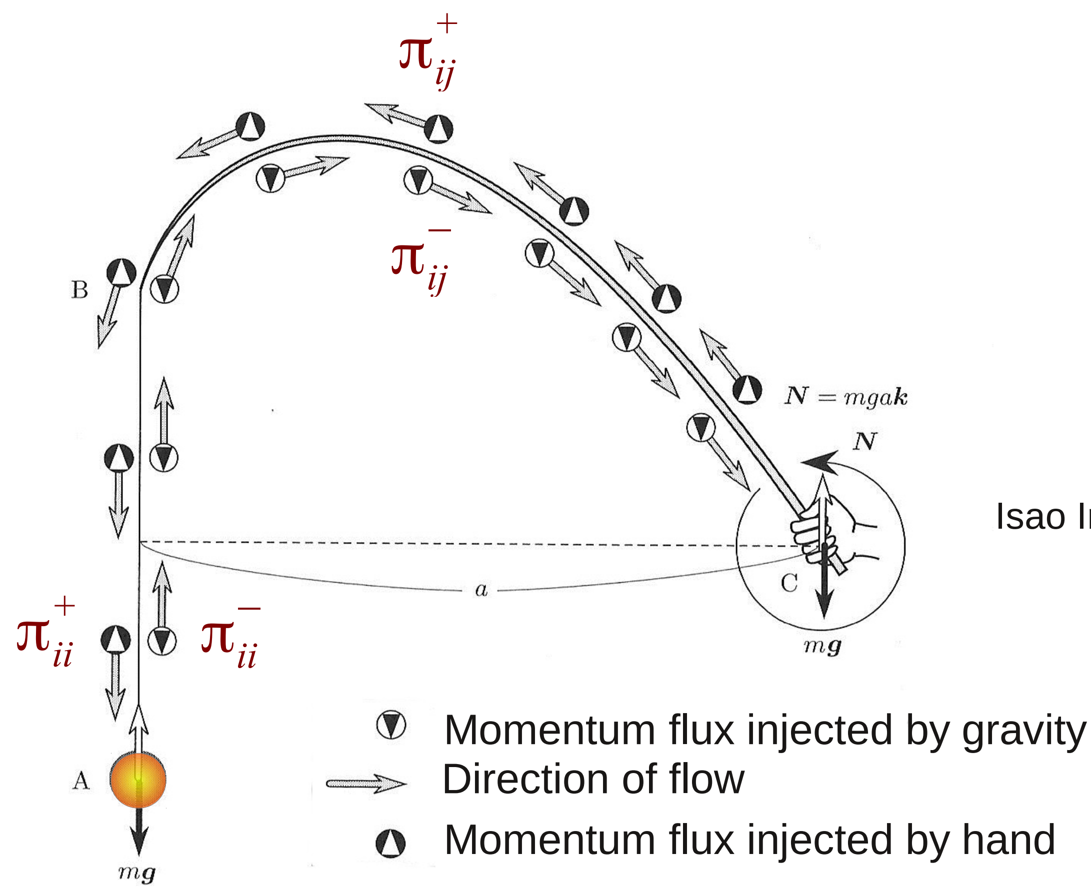
Reservoir as a heat bath





$$\kappa_1 \frac{T_H - \tilde{T}_H}{L_1} = \frac{k_B(\tilde{T}_H - \tilde{T}_C)}{M(\gamma_1^{-1} + \gamma_2^{-1})} = \kappa_2 \frac{\tilde{T}_C - T_C}{L_1}$$

Common idealization $\kappa \to \infty$ $\quad (T_H = \tilde{T}_H, T_C = \tilde{T}_C)$



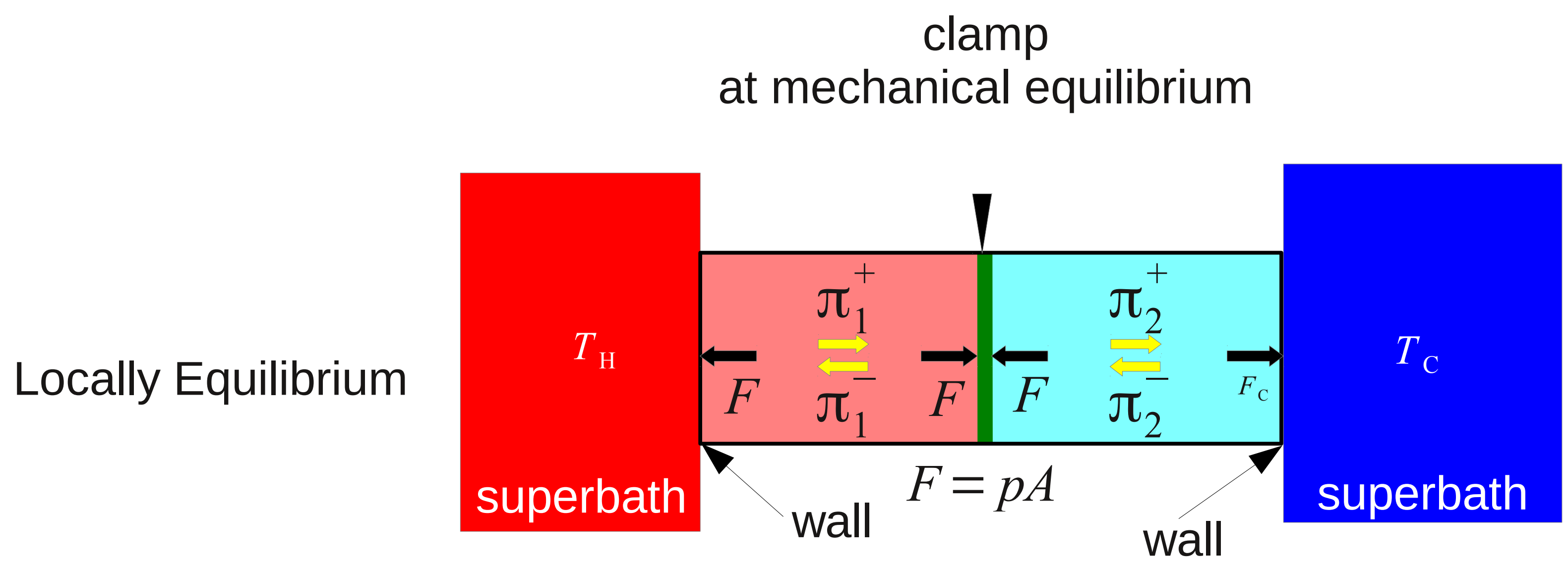
Isao Imai (2003)

$$\pi_{ii} = \pi_{ii}^{+} + \pi_{ii}^{-} = 2\pi_{ii}^{+} = \frac{-mg}{A} \qquad \left(\pi_{ii}^{+} = \pi_{ii}^{-}\right) \quad \text{Equilibrium}$$

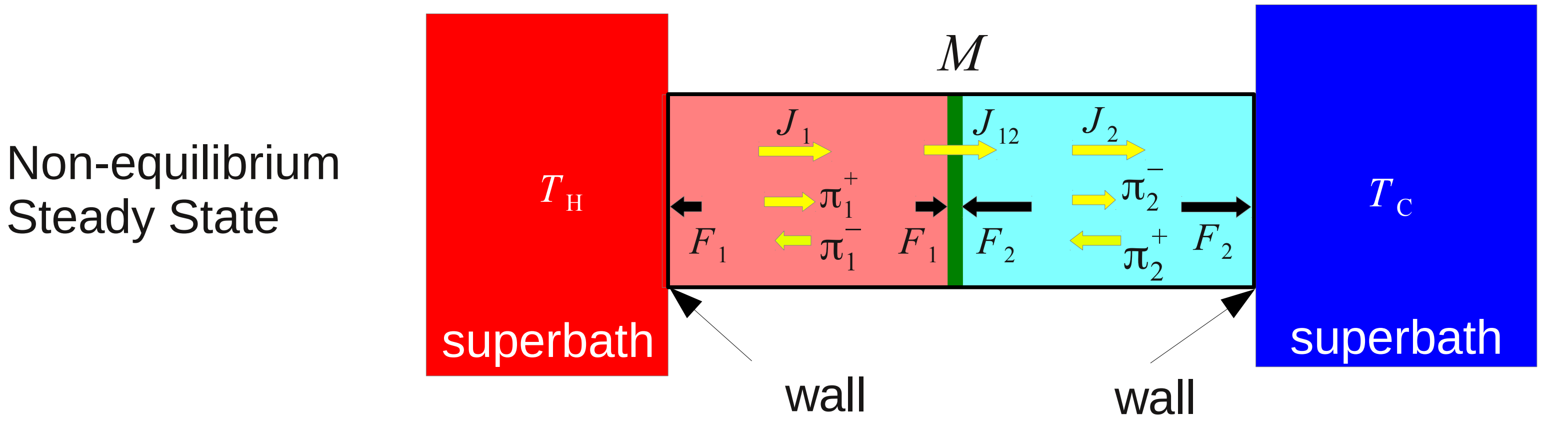
$$\pi_{ii} = \pi_{ii}^{+} + \pi_{ii}^{-} = 2\pi_{ii}^{+} - \left(\pi_{ii}^{+} - \pi_{ii}^{-}\right) \qquad \left(\pi_{ii}^{+} \neq \pi_{ii}^{-}\right)$$

Momentum Deficit

What is happening in the reservoir?



$p = n k_B T$



Environment must provides both energy and momentum flux that match to the MDD at the piston.

Momentum Deficit due to Dissipation in Hydrodynamics

$$\rho = \int m\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c}$$

mass density

$$\rho\, v_i = \int m\, c_i\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c} = 0$$

momentum density

$$p = \frac{1}{3}\mathrm{Tr}\,\pi = n\,k_{\mathrm{B}}T$$

$$\rho\, u = \int \frac{m}{2}\boldsymbol{c}\cdot\boldsymbol{c}\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c} = \frac{3}{2}n\,k_{\mathrm{B}}T$$

energy density

$$\tilde{\pi}_{ij} = \pi_{ij} - p\,\delta_{ij}$$

$$\pi_{ij} = \int m\, c_i c_j\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c}$$

pressure tensor (momentum flux)

$$q_i = \int \frac{m}{2}\boldsymbol{c}\cdot\boldsymbol{c}\, c_i\, f(\boldsymbol{x},\boldsymbol{x})\,\mathrm{d}\boldsymbol{c}$$

heat flux

$$f(\boldsymbol{x},\boldsymbol{c}) = n\left(\frac{\beta}{\pi}\right)^{3/2} \mathrm{e}^{-\beta\,\boldsymbol{c}\cdot\boldsymbol{c}} \left[1 + \frac{\beta}{p}\left\{\boldsymbol{c}\cdot\left(\tilde{\pi} + 4\frac{\beta}{5}\boldsymbol{q}\otimes\boldsymbol{c}\right)\cdot\boldsymbol{c} - 2\,\boldsymbol{q}\cdot\boldsymbol{c}\right\}\right]$$

$$\beta \equiv \frac{m}{2\,k_{\mathrm{B}}T}$$

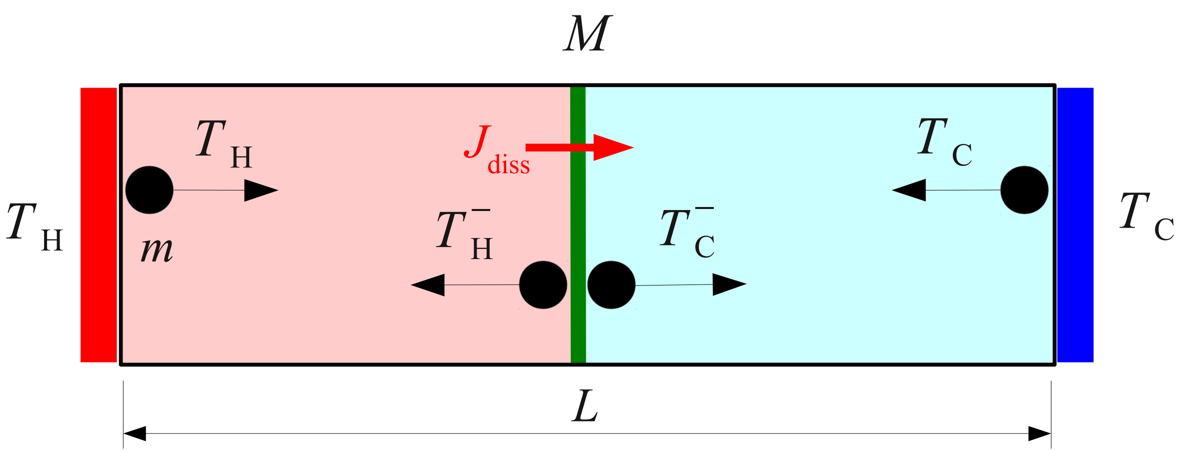
$$\pi_{ii}^{+} = \int_{c_i>0} m\, c_i c_i\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c} = \frac{\pi_{ii}}{2} + \frac{1}{5}\sqrt{\frac{2}{\pi}}\frac{q_i}{v_{\mathrm{th}}}$$

$$\pi_{ii}^{-} = \int_{c_i<0} m\, c_i c_i\, f(\boldsymbol{x},\boldsymbol{c})\,\mathrm{d}\boldsymbol{c} = \frac{\pi_{ii}}{2} - \frac{1}{5}\sqrt{\frac{2}{\pi}}\frac{q_i}{v_{\mathrm{th}}}$$

$$\pi_{ii}^{-} - \pi_{ii}^{+} = -\frac{1}{5}\sqrt{\frac{8}{\pi}}\frac{q_i}{v_{\mathrm{th}}}$$

$$F_{\mathrm{MDD}} \approx -c\,\frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

Knudsen gas $L < \lambda$



$$f_j(v) = \begin{cases} \rho_j^+ \sqrt{\dfrac{2m}{\pi k_B T_j}}\, e^{-m v^2 / k_B T_j}, & v = \text{to piston} \\ \rho_j^- \sqrt{\dfrac{2m}{\pi k_B T_j^-}}\, e^{-m v^2 / k_B T_j^-}, & v = \text{from piston} \end{cases} \qquad j = \mathrm{C}, \mathrm{H}$$

$$\rho_j^+ + \rho_j^- = \rho_j$$

$$T_j^- \sim T_j + \frac{4\epsilon}{(1+\epsilon)^2}\left(T_{\text{kin}} - T_j\right) \qquad \epsilon = \frac{m}{M}$$

Steady state conditions:

$$J^{\text{mass}} = 0$$

$$J_j^{\mathrm{E}} = J_{\text{diss}} = \text{constant}$$

$$\Rightarrow \quad \rho_j^\pm = \frac{\rho_j}{2} \pm \Delta_\rho \epsilon, \qquad T_j^- = T_j + \Delta_T \epsilon$$

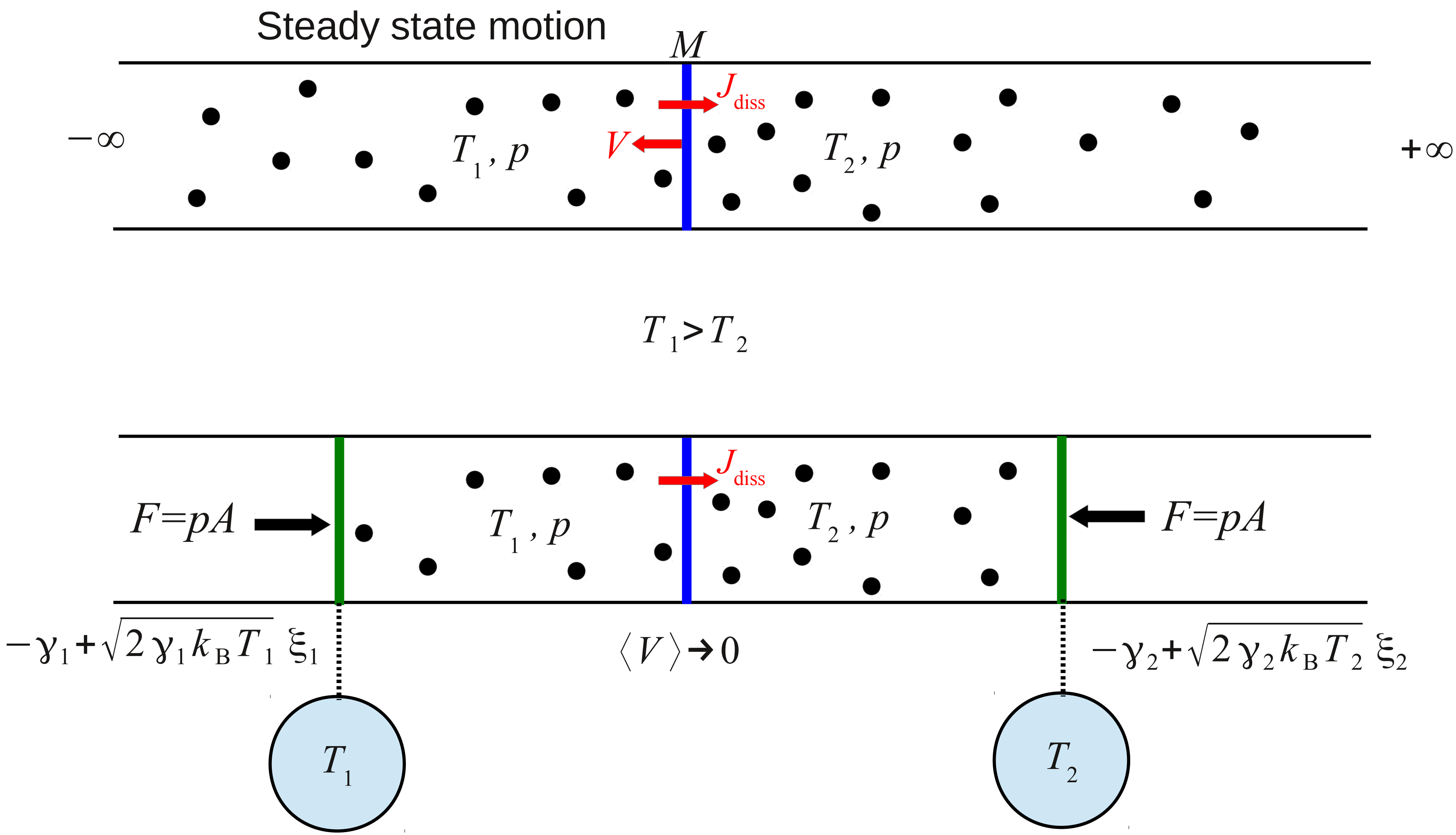
$$J_j^{\mathrm{P}} = \pi_j^+ + \pi_j^- = \rho_j^+ k_B T_j + \rho^- k_B T_j^-$$

$$\frac{F_{\text{MDD}}}{A} = J_{\mathrm{C}}^{\mathrm{P}} - J_{\mathrm{P}}^{\mathrm{P}} = \rho_{\mathrm{C}} k_B T_{\mathrm{C}} - \rho_{\mathrm{H}} k_B T_{\mathrm{H}} - \sqrt{\frac{\pi}{8}}\, J_{\text{diss}} \left(\frac{1}{v_{\mathrm{H}}^{\text{th}}} + \frac{1}{v_{\mathrm{C}}^{\text{th}}} \right)$$

Contents

- Introduction: Langevin force and Stochastic Thermodynamics
- Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.
  - Examples:
    - Brownian ratchet, Adiabatic piston,
    - Inelastic Piston, Granular ratchet
- Momentum Deficit due to Dissipation (MDD) is introduced.
- Force caused by MDD explains it all.
- The environment adjusts itself to MDD and it is no longer equilibrium.
- Sasa's Paradox

Sasa's paradox of adiabatic piston



$$\dot{P} = -\gamma_1 \langle V_1 \rangle - \gamma_2 \langle V_2 \rangle \to -(\gamma_1 + \gamma_2) \langle V \rangle$$

Conclusions

- Concept of Momentum Deficit due to Dissipation (MDD)
  Is introduced.
- Force by MDD $$F_{\mathrm{MDD}} = -c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$
- MDD captures the asymmetry in fluctuation.
- Adiabatic piston, Brownian ratchets, Inelastic piston, and
  Granular ratchet ... can be all intuitively and quantitatively explained
  by MDD without lengthy calculation.
- The environment adjusts itself to MDD and becomes non-equilibrium.
- We don't know a simple stochastic model for such an environment.
  (Does it exist?)